Math 571: Model Theory
Spring Semester 2008
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Some exercises for sections 12 and 13
Also see all the exercises that are implicit in the lecture notes, especially the "Facts".

12.1. Let $\mathcal{A}, \mathcal{B}$ be L-structures that are elementarily equivalent. Show that there exist elementary extensions $\mathcal{A}'$ of $\mathcal{A}$ and $\mathcal{B}'$ of $\mathcal{B}$ such that $\mathcal{A}' \cong \mathcal{B}'$.

12.2. Let $\mathcal{A}, \mathcal{B}$ be L-structures and let f be a nonempty elementary map from a subset of A into B. Show that there exist elementary extensions $\mathcal{A}'$ of $\mathcal{A}$ and $\mathcal{B}'$ of $\mathcal{B}$ and an isomorphism g of $\mathcal{A}'$ onto $\mathcal{B}'$ such that g is an extension of f.

12.3. Let L be the language whose nonlogical symbols consist of infinitely many constant symbols $\{c_n \mid n \in \mathbb{N}\}$. Let T be the L-theory whose axioms are $c_m \neq c_n$ for all distinct $m, n \in \mathbb{N}$. It follows from Example 3.16(ii) that T admits QE. Every model of T has a substructure isomorphic to $(\mathbb{N}, n)_{n\in\mathbb{N}}$, so T is complete by Corollary 5.5(2).
• Which countable model of T is ω-saturated?
• Which countable models of T are strongly ω-homogeneous.

12.4. Let L be the language whose nonlogical symbols consist of a unary function symbol F. Let T be the theory in L of the class of all L-structures (A, f) in which f is a bijection from A onto itself and f has no finite cycles. From Problem 2.2 we know that T admits QE and is complete. From Exercise 10.2 we know that T is strongly minimal and we understand the meaning of the dimension of a model of T. Note that $(\mathbb{Z}, S)$ is a model of T, where $S(a) = a + 1$ for all $a \in \mathbb{Z}$; therefore $T = \mathrm{Th}(\mathbb{Z}, S)$ and this model of T obviously has dimension 1.
• Which countable models of T are strongly ω-homogeneous?

12.5. Let $\mathcal{A}$ be an L-structure and $B \subseteq A$. Recall that $R \subseteq A^m$ is called *B-definable in $\mathcal{A}$* if there is an L-formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ and parameters $b_1, \ldots, b_n$ from B such that

$$R = \{(a_1, \ldots, a_m) \in A^m \mid \mathcal{A} \models \varphi[a_1, \ldots, a_m, b_1, \ldots, b_m]\}.$$

Now suppose $\mathcal{A}$ is κ-saturated and strongly κ-homogeneous and $B \subseteq A$ has $\mathrm{card}(B) < \kappa$. Suppose further that $R \subseteq A^m$ is A-definable in $\mathcal{A}$.
• Show that R is B-definable in $\mathcal{A}$ iff R is fixed setwise by every automorphism of $\mathcal{A}$ that fixes B pointwise.

13.1. Let T be a complete L-theory and let Σ be a partial n-type in L. If T has a model that omits Σ, show that Σ is locally omitted by T.

13.2. Let T be a complete L-theory and $p \in S_n(T)$. Show that p is locally omitted by T iff p is not an isolated point in the compact space $S_n(T)$.

13.3. Let T be a complete theory in a countable language, with no finite models. Show that T has a countable atomic model iff for each $n \geq 1$ the set of isolated points is dense in the space $S_n(T)$.

13.4. Let T be a complete theory in a countable language. For each positive integer k let $\Sigma_k(\bar{x})$ be a partial n_k-type in L that is omitted in some model $\mathcal{A}_k$ of T. Show that there is a single countable model $\mathcal{A}$ of T that omits $\Sigma_k(\bar{x})$ for all k.

13.5. Let L be a countable language and let L' be the result of adding countably many new predicate symbols $\{P_1, P_2, \ldots\}$ to L. Let T be a complete theory in the language L' and let $\Gamma(x_1, \ldots, x_n)$ be a set of formulas in L. Let T_m be the set of sentences in T that contain P_n only for $n = 1, \ldots, m$. Assume that for each m, T_m has a model that omits $\Gamma(x_1, \ldots, x_n)$. Show that T has a model that omits $\Gamma(x_1, \ldots, x_n)$.

13.6. Let T be a complete theory in a countable language and assume T has no finite models. Show that T is ω-categorical iff T has a countable model that is both atomic and ω-saturated.

13.7. Let T be one of the following theories. (Each is a complete theory in a countable language, with no finite models.)

+ Equality on an infinite set with infinitely many named elements. (Example 3.16(ii), Exercise 10.1)

+ Infinite vector spaces over a field K. (Exercises 3.6, 5.4, 9.3, 10.3)

+ ACF_p for a fixed characteristic p.

+ Bijections without a finite cycle. (Problem 2.2, Exercise 10.2)

+ Discrete linear orderings without endpoints. (Example 5.6, Exercises 5.3, 9.4)

+ Discrete linear orderings with minimum but no maximum. (Problem 3.1)

+ Descending equivalence relations with infinite splitting of classes. (Problem 4.1)

+ Dense linear orderings with increasing sequence of elements. (Problem 4.2)

For each of these theories, do the following:

• Show that T has a countable (infinite) atomic model.

• Try to describe the countable atomic model of T as a clear, specific mathematical structure. (According to Theorem 13.11, the countable infinite atomic model of a complete theory is unique up to isomorphism, if such a model exists.)

• For each principal n-type p that is consistent with T, try to give explicitly a complete formula contained in p.