Math 571: Model Theory
Spring Semester 2008
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Some exercises for sections 3 and 4
Also see all the exercises that are implicit in the lecture notes, especially the “Facts”.

3.1. Let $\mathcal{A}$ be an L-structure and X a nonempty subset of A. The *diagram of X in $\mathcal{A}$*, denoted by $\mathrm{Diag}_X(\mathcal{A})$, is the set of all quantifier-free $L(X)$-sentences that are true in $(\mathcal{A}, a)_{a\in X}$. Suppose X is a set of generators for $\mathcal{A}$ and $\mathcal{B}$ is another L-structure. Show that there is a 1-1 correspondence between embeddings of $\mathcal{A}$ into $\mathcal{B}$ and expansions of $\mathcal{B}$ that are models of $\mathrm{Diag}_X(\mathcal{A})$.

3.2. Let $\mathcal{A}$ be an L-structure and X a nonempty subset of A. The *elementary diagram of X in $\mathcal{A}$*, denoted by $\mathrm{EDiag}_X(\mathcal{A})$, is the set of all $L(X)$-sentences that are true in $(\mathcal{A}, a)_{a\in X}$. Suppose X is a set of generators for $\mathcal{A}$ and $\mathcal{B}$ is another L-structure. Show that there is a 1-1 correspondence between elementary embeddings of $\mathcal{A}$ into $\mathcal{B}$ and expansions of $\mathcal{B}$ that are models of $\mathrm{EDiag}_X(\mathcal{A})$.

3.3. Let I be an index set and U an ultrafilter on I. Let $(\mathcal{A}_i \mid i \in I)$ and $(\mathcal{B}_i \mid i \in I)$ be families of L-structures. If $\mathcal{A}_i$ can be elementarily embedded in $\mathcal{B}_i$ for all $i \in I$, show that $\Pi_U \mathcal{A}_i$ can be elementarily embedded in $\Pi_U \mathcal{B}_i$.

3.4. Let $\mathcal{A}$ be an infinite L-structure and κ an infinite cardinal. Show that there exists an ultrapower of $\mathcal{A}$ that has cardinality at least κ. (Compare Corollary 1.12.) It follows that $\mathcal{A}$ has an elementary extension of cardinality at least κ.

3.5. Let $\mathcal{A} \subseteq \mathcal{B}$ be L-structures. Suppose that for every finite sequence $a_1, \dots, a_m \in A$ and every $b \in B$ there is an automorphism of $\mathcal{B}$ which fixes each element of $a_1, \dots, a_m$ and moves b into A. Show that $\mathcal{A} \preceq \mathcal{B}$.

3.6. Let K be a field and let L be the first order language of vector spaces over K; the nonlogical symbols of L are a constant 0, a binary function symbol $+$, and a unary function symbol F_a for each $a \in K$. Given a K-vector space V, we regard V as an L-structure in the obvious way: 0 is interpreted by the identity element of V, $+$ is interpreted by the addition of V, and each F_a is interpreted by the operation of scalar multiplication by a. Suppose $W \subseteq V$ are infinite dimensional K-vector spaces. Use the previous exercise to prove that $W \preceq V$. Use this result to show that any two infinite K-vector spaces are elementarily equivalent.

3.7. Let L be the language whose only nonlogical symbol is a binary predicate symbol $<$. Let $\mathcal{A}$ be an L-structure that is a dense linear ordering without endpoints. Let $\varphi(x, y_1, \dots, y_n)$ be any L-formula (with x a single variable) and let $a_1, \ldots, a_n \in A$. Show that the definable set

$$\{a \in A \mid \mathcal{A} \models \varphi[a, a_1, \ldots, a_n]\}$$

is the union of a finite number of open intervals (whose endpoints are in A) and a finite subset of A.

3.8. Let L be the pure language of $=$, so L has no nonlogical symbols, and let σ be any L-sentence. Show that if σ is satisfiable, then σ is true in some finite set.

4.1. Show that the linear ordering $(\mathbb{R}, <)$ is ω-saturated but not ω_1-saturated. (Note that $(\mathbb{R}, <) \models DLO$ so you can use Example 3.15.)

4.2. Show that no infinite well ordering is ω-saturated.

4.3. Let I be a countable infinite set and U a nonprincipal ultrafilter on I.

• Let $\mathcal{A}$ be the linear ordering $(\mathbb{Q}, <)$. Show that the cardinality of the ultrapower $\mathcal{A}^I/U$ is exactly 2^ω. (Note that it is not enough to prove that the ultraproduct is uncountable; it is possible that $\omega_1 < 2^\omega$.)

• More generally, let L be any first order language, and let $\mathcal{A}_i$ be a countable infinite L-structure for each $i \in I$. Show that the cardinality of the ultraproduct $\Pi_U(\mathcal{A}_i \mid i \in I)$ is exactly 2^ω.