Problem 1 (6 points)

(Multiple choice) The following two subproblems are independent of each other. For each problem exactly one of the answers is correct. Circle the correct answer. You will get no credit if you circle more than one answer, but there is no penalty for checking the wrong answer (so if you don't know the answer, you can't lose if you pick one at random).

(i) The pictured surface represents

(a) a hyperboloid with one sheet

(b) a hyperboloid with two sheets

(c) an elliptic paraboloid

(d) a hyperbolic paraboloid

(e) a cone

(f) an ellipsoid

Solution: (d) a hyperbolic paraboloid (since it has the saddle shape characteristic of a hyperbolic paraboloid)

(ii) The equation $x^2 + y^2 - z^2 = 0$ represents a

(a) a hyperboloid with one sheet

(b) a hyperboloid with two sheets

(c) an elliptic paraboloid

(d) a hyperbolic paraboloid

(e) a cone

(f) an ellipsoid

Solution: (e) a cone (The equation is equivalent to $z^2 = x^2 + y^2$, which is that of a circular cone. Note that it can't be a hyperboloid because there is no constant term in the equation—the right-hand side is 0, not 1.)

Problem 2 (12 points)

(Multiple choice) The following subproblems are independent of each other. For each problem exactly one of the answers is correct. Circle the correct answer. You will get no credit if you circle more than one answer, but there is no penalty for checking the wrong answer (so if you don't know the answer, you can't lose if you pick one at random).

(i) The **component** of $\vec{a}$ along $\vec{b}$ (i.e., the projection of $\vec{a}$ onto $\vec{b}$, $\text{comp}_{\vec{b}}\,\vec{a}$) is given by

(a) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ (b) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ (c) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ (d) $\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ (e) none of the above

Solution: $\boxed{\text{(b) } \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}$

(ii) The volume of the **parallelepiped** determined by three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

(a) $|(\vec{a}\cdot\vec{b})\cdot\vec{c}|$ (b) $|(\vec{a}\times\vec{b})\times\vec{c}|$ (c) $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ (d) $|\vec{a}\times(\vec{b}\cdot\vec{c})|$ (e) none of the above

Solution: $\boxed{\text{(c) } |(\vec{a}\times\vec{b})\cdot\vec{c}|}$

(iii) If $\vec{a}$ and $\vec{b}$ are vectors forming an angle of $\pi/6$ and of lengths $|\vec{a}|=1$ and $|\vec{b}|=2$, and $\vec{c}=\vec{a}\times\vec{b}$, then the vector $\vec{c}$ has length (=magnitude)

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) 1 (e) none of the above

Solution: $\boxed{\text{(d) } 1}$ (since $|\vec{a}|\cdot|\vec{b}|\cdot\sin(\pi/6)=2\cdot1\cdot(1/2)=1$)

Comments: Partial credit was given for arriving at the formula $|\vec{a}|\cdot|\vec{b}|\cdot\sin(\pi/6)$, even if the circled answer was wrong due to an incorrect evaluation of $\sin(\pi/6)$.

(iv) If $\vec{a}$ and $\vec{b}$ are **perpendicular** vectors with lengths $|\vec{a}|=1$ and $|\vec{b}|=2$ and $\vec{c}=\vec{b}-\vec{a}$, then $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is equal to
(a) 5 (b) 4 (c) $\sqrt{5}$ (d) 2 (e) 0 (f) none of the above

Solution: $\boxed{\text{(e) } 0}$ (Since $\vec{c}=\vec{b}-\vec{a}$ lies in the same plane as $\vec{a}$ and $\vec{b}$, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, are coplanar, so the scalar triple product is 0. Alternatively, one can argue as follows: Since $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$, we have $(\vec{a}\times\vec{b})\cdot\vec{a}=0$ and $(\vec{a}\times\vec{b})\cdot\vec{b}=0$ and thus $(\vec{a}\times\vec{b})\cdot(\vec{b}-\vec{a})=0-0=0$.

Problem 3 (12 points)

Let L_1 be the line that is perpendicular to the plane $x + 2y + 3z = 0$ and passes through the point $(2, -1, 3)$, and let L_2 be the line given by the symmetric equations $\frac{1}{3}(x - 5) = \frac{1}{2}(y - 1) = z - 4$.

(i) Find parametric equations for the line L_1.

Solution: A normal vector to the given plane is $\overrightarrow{n} = \langle 1, 2, 3\rangle$ (read off the coefficients of x, y, and z). Since L_1 is perpendicular to this plane, this normal vector is a direction vector for L_1. Thus, a vector equation for L_1 is $\overrightarrow{r} = \langle 2, -1, 3\rangle + t\langle 1, 2, 3\rangle$, and parametric equations are

$$\boxed{x = 2 + t, \quad y = -1 + 2t, \quad z = 3 + 3t}.$$

(ii) Find parametric equations for the line L_2.

Solution: A direction vector for this line is $\overrightarrow{v} = \langle 3, 2, 1\rangle$ (read off the denominators in the expressions involving x, y, and z, in the symmetric equations). A point on the line can be obtained, for example, by setting all three terms in the symmetric equation equal to 0 and solving for x, y, and z, giving $x = 5$, $y = 1$, $z = 4$. Using this point along with the above direction vector we get the vector equation $\overrightarrow{r} = \langle 5, 1, 4\rangle + t\langle 3, 2, 1\rangle$ and the parametric equations

$$\boxed{x = 5 + 3t, \quad y = 1 + 2t, \quad z = 4 + t}.$$

(iii) Determine whether the lines L_1 and L_2 are parallel, skew, or intersect. If they intersect, determine the intersection point.

Solution: The direction vectors for the two lines are $\langle 1, 2, 3\rangle$ and $\langle 3, 2, 1\rangle$. These vectors are (obviously) not multiples of each, so the lines are not parallel. To determine whether they intersect or are skew, we equate the x, y and z values in the two parametric equations (using different notations for the two parameters, say t for the first and s for the second) and check whether the resulting system of equations has a solution. The system of equations we get is

$$\begin{aligned} &(1) \quad 2 + t = 5 + 3s, \\ &(2) \quad -1 + 2t = 1 + 2s, \\ &(3) \quad 3 + 3t = 4 + s. \end{aligned}$$

From (1) we get $t = 3 + 3s$. Substituting this into (2) gives $-1 + 2(3 + 3s) = 1 + 2s$. Solving for s we get $\boxed{s = -1}$. Therefore, by (1), $\boxed{t = 0}$. Substituting these values ($s = -1$ and $t = 0$) into (3), we see that (3) is also satisfied, so the above system indeed has a solution. Hence $\boxed{\text{the two lines intersect}}$. The intersection point is obtained by plugging $t = 0$ into the parametric equations for L_1: $(x, y, z) = \boxed{(2, -1, 3)}$.

Comments: An alternative solution is to leave L_2 in symmetric equation form and plug the parameter equations for L_1 into these equations. This gives $\frac{1}{3}((2 + t) - 5) = \frac{1}{2}((-1 + 2t) - 1) = (3 + t) - 4$, which simplifies to $\frac{1}{3}t - 1 = t - 1 = t - 1$. The latter system has solution $t = 0$, so an intersection point exists, and plugging $t = 0$ into the parametric equations for L_1 gives $(x, y, z) = (2, -1, 3)$ as intersection point.

If all this looked vaguely familiar, it's no accident; the problem is essentially Problem 5 in 11.4, one of the assigned hw problems from this section.

Problem 4 (12 points)

A moving spaceship has instruments displaying the velocity and acceleration of the spaceship (but not the position) at any given time. Suppose that, at a particular point in time (for example, at the stroke of midnight on October 31), the instruments show that the velocity of the spaceship at this point in time is $\langle 1, 2, 3\rangle$, and the acceleration $\langle 0, 1, 3\rangle$.

(i) In which direction is the spaceship moving at this point in time? (Given the direction as a **unit vector**, such as $\langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$.)

Solution: Since the velocity vector is always tangent to the path of motion, its direction is the direction in which the spaceship is moving. Thus, to get a unit vector in this direction, which in fact is just the unit tangent vector $\vec{T}$, all we need to do is normalize the velocity vector $\vec{v} = \langle 1, 2, 3\rangle$:

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 2, 3\rangle}{\sqrt{1^2 + 2^2 + 3^2}} = \boxed{\frac{1}{\sqrt{14}}\langle 1, 2, 3\rangle}$$

(ii) What is the **tangential component** of the acceleration at this point in time?

Solution: Since we are given $\vec{v}(= vrp)$ and $\vec{a}(= \vec{r}\,'')$, the appropriate formula for a_T to use is the one involving $\vec{r}\,'$ and $\vec{r}\,''$:

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}.$$

Plugging in the given values for $\vec{v}$ and $\vec{a}$, we get

$$a_T = \frac{\langle 1, 2, 3\rangle \cdot \langle 0, 1, 3\rangle}{\sqrt{1^2 + 2^2 + 3^2}} = \boxed{\frac{11}{\sqrt{14}}}.$$

Comments: Note that the other formula for a_T, $a_T = v'$, is of no use here since we don't know v'. All we know is that the velocity $\vec{v}$ is equal to $\langle 1, 2, 3\rangle$ *at a particular point in time*, so the speed $v = |\vec{v}|$ is equal to $v = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ *at this point in time*. However, since we do not have a general formula for $\vec{v}(t)$ for all times t, we cannot take derivatives to compute v'. Note that v' is *not* the same as $|\vec{a}|$.

(iii) What is the **curvature** of the flight path at this point in time?

Solution: Again we need the formula involving $\vec{r}\,'$ and $\vec{r}\,''$ for κ:

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}.$$

We get

$$\begin{aligned}
\kappa &= \frac{|\langle 1, 2, 3\rangle \times \langle 0, 1, 3\rangle|}{(\sqrt{1^2 + 2^2 + 3^2})^3} = \frac{1}{14\sqrt{14}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} \\
&= \frac{1}{14\sqrt{14}} \left| \left(\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \vec{k} \right) \right| \\
&= \frac{|\langle 3, -3, 1\rangle|}{14\sqrt{14}} = \boxed{\frac{\sqrt{19}}{14\sqrt{14}}}.
\end{aligned}$$

Comments: As in part (ii), the other formula for κ, namely $\kappa = |\vec{T}\,'|/|\vec{r}\,'|$, is of no use here, since we do not know $\vec{T}\,'$. All we know is the unit tangent vector $\vec{T}$ *at the given point in time*, but we don't have a general formula for $\vec{T}(t)$ and thus cannot compute derivatives.

Problem 5 (12 points)

A lark positioned at point $(1,0,0)$ at time $t=0$ takes off with an initial velocity $\vec{v}(0)=\langle 0,1,2\rangle$, and acceleration at time t given by $\vec{a}(t)=\langle -\cos t,-\sin t,0\rangle$. Compute the following quantities.

(i) The velocity and position of the lark at any time t.

Solution: This is a standard motion problem, as in Problem 27 of 11.5. Given the initial position $\vec{r}(0)$, the initial velocity $\vec{v}(0)$, and the acceleration $\vec{a}(t)$ at any time t, and we need to find the velocity $\vec{v}(t)$ and position $\vec{r}(t)$ at any time t. This is simply a matter of two integrations, but—as with all integrations—one has to make sure to add an integration constant (in this case, a constant *vector*) at each integration step, and then solve for these constants by inserting the given initial values $\vec{r}(0)$ and $\vec{v}(0)$.

We begin by integrating $\vec{a}(t)=\langle -\cos t,-\sin t,0\rangle$, to get

$$\vec{v}(t)=\int \vec{a}(t)dt=\langle -\sin t,\cos t,0\rangle+\vec{c_1},$$

where $\vec{c_1}$ is a constant vector. Plugging in $t=0$ and using the given initial velocity $\vec{v}(0)=\langle 0,1,2\rangle$, we get

$$\langle 0,1,2\rangle=\vec{v}(0)=\langle 0,1,0\rangle+\vec{c_1},$$

so $\vec{c_1}=\langle 0,0,2\rangle$. Thus, the velocity is at any given time t is

$$\boxed{\vec{v}(t)=\langle -\sin t,\cos t,2\rangle}$$

A second integration gives

$$\vec{r}(t)=\int \vec{v}(t)dt=\langle \cos t,\sin t,2t\rangle+\vec{c_2},$$

where $\vec{c_2}$ is a constant vector. Plugging in $t=0$ and using the given initial position $\vec{r}(0)=\langle 1,0,0\rangle$, we get

$$\langle 1,0,0\rangle=\vec{r}(0)=\langle 1,0,0\rangle+\vec{c_1},$$

so $\vec{c_1}=\langle 0,0,0\rangle$ and the position at any time t is

$$\boxed{\vec{r}(t)=\langle \cos t,\sin t,2t\rangle}.$$

Note that this is a helix spiraling upwards—exactly the kind of flight path that a lark would take!

Comments: As noted above, the key is to keep track of the integration constants arising and to properly solve for these by inserting the initial conditions. If one leaves out the integration constants entirely, one would get $\vec{v}(t)=\langle -\sin t,\cos t,0\rangle$ and $\vec{r}(t)=\langle \cos t,\sin t,0\rangle$, which describes a circling motion taking place entirely in the xy-plane; that is, the lark would not even get off the ground! Another incorrect solution is obtained by simply adding the initial vectors to the above formulas for $\vec{v}(t)$. This leads to $\vec{v}(t)=\langle -\sin t,\cos t+1,2\rangle$ and $\vec{r}(t)=\langle \cos t+1,\sin t+t,2t\rangle$, but plugging $t=0$ into the formula for $\vec{v}(t)$ gives $\vec{v}(0)=\langle 0,2,2\rangle$, which is not the given initial velocity.

(ii) The unit normal vector $\vec{N}(t)$ to the lark's path at any time t.

Solution: The unit normal vector is given by $\vec{N}(t)=\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$. We have

$$\begin{aligned}
\vec{T}(t)&=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\frac{1}{\sqrt{5}}\langle -\sin t,\cos t,2\rangle,\\
\vec{T}'(t)&=\frac{1}{\sqrt{5}}\langle -\cos t,-\sin t,0\rangle\\
|\vec{T}'(t)|&=\frac{1}{\sqrt{5}}\sqrt{(-\cos t)^2+(-\sin t)^2+0^2}=\frac{1}{\sqrt{5}},
\end{aligned}$$

so

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \boxed{\langle -\cos t, -\sin t, 0\rangle}$$

(iii) The distance the lark has to fly (i.e., the length of the flight path) in order to reach a height of 100 (i.e., a position with z-coordinate equal to 100)

Solution: The z-component of $\vec{r}(t)$ is $2t$, so height 100 is reached at time $t = 50$. Now,

$$|\vec{r}\,'(t)| = |\langle \sin t, \cos t, 2\rangle| = \sqrt{(\sin t)^2 + (\cos t)^2 + 2^2} = \sqrt{5}.$$

Hence, by the arclength formula, the distance travelled between times $t = 0$ and $t = 50$ is given by

$$L = \int_0^{50} |\vec{r}\,'(t)| dt = \int_0^{50} \sqrt{5} dt = \boxed{50\sqrt{5}}.$$

Comments: The problem did *not* ask for the *straightline* distance between the point reached at height 100 and the origin (after all, the bird chosen for this problem was a lark, not a crowe!), but rather the *length of the flight path*. The former would be simply $|\vec{r}(50)|$, but the latter is an arclength, and it has to be computed via the arclength formula.

Problem 6 (6 points)

As usual, let $\overrightarrow{r}(t)$, $\overrightarrow{v}(t)$, and $\overrightarrow{a}(t)$ denote, respectively, the position, velocity, and acceleration, of a moving particle at time t, and let

$$f(t) = \overrightarrow{r}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t)).$$

Compute $f'(t)$, **simplifying as much as possible and explaining all steps of your work** (e.g., if you drop a term, state why you can do that).

Solution: This is an exercise in applying differentiation rules, and properties of dot and cross products, especially with regard to parallel and perpendicular vectors. The steps required in the solution are the following:

1. Apply the product rule for dot products to get

$$\begin{aligned} f'(t) &= \overrightarrow{r}\,'(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t)) + \overrightarrow{r}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t))' \\ &= \overrightarrow{v}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t)) + \overrightarrow{r}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t))'. \end{aligned} \tag{1}$$

2. Observe that the term $\overrightarrow{v}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t))$ is equal to 0, **since** $\overrightarrow{v}(t) \times \overrightarrow{a}(t)$ **is perpendicular to** $\overrightarrow{v}(t)$ **and the dot product of perpendicular vectors is zero**. Thus, (1) simplifies to

$$f'(t) = \overrightarrow{r}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}(t))'. \tag{2}$$

3. Apply the product rule for cross products again to get

$$\begin{aligned} (\overrightarrow{v}(t) \times \overrightarrow{a}(t))' &= \overrightarrow{v}\,'(t) \times \overrightarrow{a}(t) + \overrightarrow{v}(t) \times \overrightarrow{a}\,'(t). \\ &= \overrightarrow{a}(t) \times \overrightarrow{a}(t) + \overrightarrow{v}(t) \times \overrightarrow{a}\,'(t). \end{aligned} \tag{3}$$

4. Observe that the term $\overrightarrow{a}(t) \times \overrightarrow{a}(t)$ is equal to 0, **since the vector** $\overrightarrow{a}(t)$ **is parallel to itself and the cross product of parallel vectors is zero**. Thus, (3) simplifies to

$$(\overrightarrow{v}(t) \times \overrightarrow{a}(t))' = \overrightarrow{v}(t) \times \overrightarrow{a}\,'(t). \tag{4}$$

5. Combine (2) and (4) to get the final formula,

$$\boxed{f'(t) = \overrightarrow{r}(t) \cdot (\overrightarrow{v}(t) \times \overrightarrow{a}\,'(t))}.$$

(The explanations sought are in boldface.)

Comments: This is similar to Problem 42 in 11.5, a hw problem, which was also discussed in the recitation sections. The main point of such problems is that, after applying the product rule for dot/cross products, certain terms drop out due to properties of the dot/vector product, and after these simplifications the resulting expression reduces to a single term. In the above problem, there are two such simplifications, given in steps 2 and 4. *To get credit on this problem, these two simplifications have to be carried out, with appropriate explanations.* (Partial credit was given if only one of the above simplifications was carried out, assuming everything else was correct.)