UIUC Mock Putnam Exam (Elementary Version)
1/2003
Solutions

Problem 1. How many positive integers are there whose binary expansion has *exactly* 2003 digits and the sum of its binary digits is even.

Solution. Exactly 2003 digits implies that the first digit is 1, so we have 2002 remaining binary digits whose sum is odd. The number is equal to the number of subsets of a set with 2002 elements with odd number of elements, which is 2^{2001}.

Problem 2. Can 2003 be expressed as a sum of the squares of two integers?

Solution. No. Assume that there are integers x, y such that $x^2+y^2 = 2003$. Then one of them must be even, the other one odd, say $x = 2k$, $y = 2m+1$. Then x^2 is divisible by 4 (that is $x^2 \equiv 0 \pmod 4$), while $y^2 = 4(m^2+m)+1$, thus $y^2 \equiv 1 \pmod 4$. So, $x^2 + y^2 \equiv 1 \pmod 4$, while $2003 \equiv 3 \pmod 4$, which is a contradiction.

Problem 3. If n is a positive integer, count the solutions in nonnegative integers $x_1, \ldots, x_k$ to the equation $x_1 + \cdots + x_k \leq n$.

Solution. One way to solve the problem is to count the solutions to $x_1 + \cdots + x_k = m$, which is $\binom{m+k-1}{k-1}$. Then sum over $0 \leq m \leq n$ and apply a summation identity to obtain $\binom{n+k}{k}$ solutions. A faster way is to introduce a new variable x_{k+1} so that the sum of all $k+1$ variables is equal to n. Thus the number of solutions is again $\binom{n+k}{k}$. This is also a way to prove the above summation identity.

Problem 4. In how many ways can one distribute 100 slices of pizza between 20 students if every student gets at least one slice?

Solution. Solutions in positive integers to $x_1 + \cdots + x_{20} = 100$ correspond to solutions in nonnegative integers to $y_1 + \cdots + y_{20} = 100 - 20$. Therefore, the number of solutions is $\binom{80+(20-1)}{20-1} = \binom{99}{19}$.

Problem 5. Evaluate the integral

$$I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

Solution. Denote $r = \sqrt{2}$. Change the variable $x = \pi/2 - u$. Then,

$$I = \int_{\pi/2}^{0} \frac{-du}{1 + \cot^r u} = \int_0^{\pi/2} \frac{\tan^r u du}{\tan^r u + 1}.$$

Then, adding the two representations, $2I = \int_0^{\pi/2} 1 dx = \pi/2$. Thus, $I = \pi/4$.