UIUC Mock Putnam Exam 2/2003

10-29-03

Problem 1. Evaluate

$$\sum_{n=0}^{\infty}(n+4)3^{-n}.$$

Solution. The value of the sum is $\boxed{27/4}$.

Start with the series expansion

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Differentiating term by term gives

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n-1}.$$

Thus

$$\frac{x}{(1-x)^2}+\frac{4}{1-x}=\sum_{n=0}^{\infty}(n+4)x^n$$

and so

$$\sum_{n=0}^{\infty}\frac{n+4}{3^n}=\frac{1/3}{(1-1/3)^2}+\frac{4}{1/1/3}=\frac{27}{4}.$$

Problem 2. Find the volume of the solid region in space which is the intersection of the three open cylinders $\{x^2+y^2<1\}$, $\{x^2+z^2<1\}$ and $\{y^2+z^2<1\}$.

Solution. The volume in question is $\boxed{8\sqrt{2}(\sqrt{2}-1)}$.

Set $C_1=\{x^2+y^2<1\}$, $C_2=\{x^2+z^2<1\}$ and $C_3=\{y^2+z^2<1\}$. Introducing the usual cylindrical coordinates r,θ,z, we have $C_1=\{r<1\}$, $C_2=\{r^2\cos^2\theta+z^2<1\}$ and $C_3=\{r^2\sin^2\theta+z^2<1\}$, whence $C_1\cap C_2\cap C_3=\{r<R(z,\theta)\}$, where

$$R(z,\theta):=\min\left\{1,\frac{\sqrt{1-z^2}}{|\sin\theta|},\frac{\sqrt{1-z^2}}{|\cos\theta|}\right\}.$$

It follows that the volume in question is equal to

$$V = \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{R(z,\theta)} r\, dr\, d\theta\, dz = 8 \int_{0}^{1} \int_{0}^{\pi/4} R(z,\theta)^2\, d\theta\, dz.$$

For $0 < \theta < \pi/4$ we have $R(z,\theta) = 1$ if $0 < z < \sin\theta$ and $R(z,\theta) = \sqrt{1-z^2}/\cos\theta$ if $\sin\theta < z < 1$. It follows that

$$\begin{aligned} V &= 8 \int_{0}^{\pi/4} \left(\sin\theta + \int_{\sin\theta}^{1} \frac{1-z^2}{\cos^2\theta}\, dz \right) d\theta \\ &= 8 \int_{0}^{\pi/4} \left(\sin\theta + \frac{\frac{2}{3} - \frac{2}{3}\sin\theta - \frac{1}{3}\sin\theta\cos^2\theta}{\cos^2\theta} \right) d\theta \\ &= \frac{16}{3} \int_{0}^{\pi/4} (\sin\theta + \sec^2\theta - \sec\theta\tan\theta)\, d\theta \\ &= \frac{16}{3} (-\cos\theta + \tan\theta - \sec\theta)\Big|_{0}^{\pi/4} = 8\sqrt{2}(\sqrt{2}-1). \end{aligned}$$

Problem 3. Show (without using a calculator or doing extensive computation) that

$$\log_{2003} 2004 + \log_{2004} 2003 > 2$$

($\log_a b$ denotes the base a logarithm of b).

Solution. By properties of the logarithm, the stated inequality is equivalent to

$$\frac{\log 2004}{\log 2003} + \frac{\log 2003}{\log 2004} > 2.$$

Set $x = \log 2004/\log 2003$; we must show that

$$x + \frac{1}{x} > 2. \tag{1}$$

In fact, (1) holds for any value $x > 0$, $x \neq 1$, as is easily established with elementary calculus. Set $f(x) = x + 1/x$; then $f'(x) = 1 - 1/x^2$. Thus for $x > 0$ we have $f'(x) = 0$ if and only if $x = 1$. Furthermore, $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $0 < x < 1$ which implies that for all $x > 0$, $x \neq 1$, $f(x) > f(1) = 2$.

Problem 4. For $a \geq 2$ evaluate the integral

$$\int_{0}^{\infty} \frac{1}{a^2 + (x - \frac{1}{x})^2}\, dx.$$

Solution. The value of the integral is $\boxed{\pi/(2a)}$.

Method I: The integral in question is

$$I = \int_{0}^{\infty} \frac{x^2\, dx}{x^4 + (a^2-2)x^2 + 1}. \tag{2}$$

Making the change of variables $u = 1/x$ shows that

$$I = \int_0^\infty \frac{du}{u^4 + (a^2-2)u^2 + 1}. \tag{3}$$

Since $a \geq 2$, $x^4 + (a^2-2)x^2 + 1 = (x^2+b^2)(x^2+c^2)$ for positive real numbers b, c. Then $b^2 + c^2 = a^2 - 2$, $bc = 1$, and

$$\frac{1}{x^2+b^2} + \frac{1}{x^2+c^2} = \frac{2x^2 + a^2 - 2}{x^4 + (a^2-2)x^2 + 1}.$$

Using (2) and (3) we compute

$$\int_0^\infty \frac{dx}{x^2+b^2} + \int_0^\infty \frac{dx}{x^2+c^2} = 2I + (a^2-2)I = a^2 I.$$

On the other hand, the left hand side of this identity is equal to

$$\frac{\pi}{2b} + \frac{\pi}{2c} = \frac{\pi(b+c)}{2bc}.$$

Since $b, c > 0$ we have $b + c = \sqrt{b^2 + 2bc + c^2} = a$ so $a^2 I = a\pi/2$ as desired.

Method II: Let $f(t) = 1/(1+t^2)$ and

$$F(y) = \int_0^\infty f(x - \frac{y}{x})\, dx.$$

Thus the integral in question is $I = F(1)$. We will show that $F'(y) = 0$ so that $F(y)$ is a constant. Since $F(0) = \int_0^\infty 1/(a^2+x^2)\, dx = \pi/(2a)$ this will show that $I = \pi/(2a)$ as desired.

Differentiating under the integral sign gives

$$F'(y) = \int_0^\infty f'(x - \frac{y}{x})(-\frac{1}{x})\, dx. \tag{4}$$

Making the change of variables $u = y/x$ gives

$$F'(y) = \int_0^\infty f'(\frac{y}{u} - u)(-\frac{1}{u})\, du.$$

Since f is an even function, f' is odd and so

$$F'(y) = \int_0^\infty f'(u - \frac{y}{u})(\frac{1}{u})\, du.$$

Comparing with (4), we see that $F'(y) = -F'(y)$ and so $F'(y) = 0$.