UIUC Department of Mathematics

Mock Putnam Exam 1 Solutions

October 5, 1998

1. Find the smallest positive integer n such that every digit of $15n$ is either 8 or 0.

Solution. Let $N = 15n$. Since the number N is divisible by 3 and 5, the sum of its digits must be divisible by 3, and the last digit must be 0 or 5. If N consists only of digits 0 and 8, it follows that the last digit must be 0 and the number of digits 8 contained in N must be a multiple of 3. The smallest positive number with these properties is $N = 8880$, so $n = N/15 = 592$ is the smallest positive integer such that every digit of $15n$ is 8 or 0.

2. Water is poured from a cylindrical container of radius 1" and height 1" by slowly tilting the container to one side. What is the volume of water remaining when exactly half of the base of the container still is emerged in water?

Solution. Imagine that the water in the cylinder has frozen at the point where exactly half of the base was still submerged, and the cylinder has been placed back on its base. The part of the cylinder that was originally filled with water and which is now filled with ice then represents a 3-dimensional solid with "base" given by $z = 0$, $x \geq 0$, and $x^2+y^2 \leq 1$ (a semi-circle in the xy-plane) and with "roof" $z = x$. The volume in question is the volume V of this solid. To compute V, we divide the solid into vertical slabs of width ϵ, add up the approximate volumes of these slabs, and let $\epsilon \to 0$. To this end, pick a large integer N, let $\epsilon = 1/N$, and slice the solid by the vertical planes $x = 0$, $x = \epsilon$, $x = 2\epsilon, \ldots, x = N\epsilon$. The slab between the planes $x = k\epsilon$ and $x = (k+1)\epsilon$ has height approximately $k\epsilon$, and the area of the base is approximately $2\epsilon\sqrt{1-(k\epsilon)^2}$. Thus, the volume of this slab is about $2(k\epsilon^2)\sqrt{1-(k\epsilon)^2}$. Adding up these volumes for $k = 0, 1, \ldots, N-1$ we obtain

$$V \approx \sum_{k=0}^{N-1} 2(k\epsilon^2)\sqrt{1-k^2\epsilon^2}.$$

where $\epsilon = 1/N$. The last sum is a Riemann sum for the integral $I = 2\int_0^1 x\sqrt{1-x^2}dx = 2/3$, and it therefore converges to that integral when we let $N \to \infty$. Hence $V = 2/3$. (The above argument conveys the idea behind the proof, but for a completely rigorous argument one would need to make the various "approximate" equalities precise, for example, by stating explicit lower and upper bounds and showing that the resulting lower and upper bounds for V converge to the same limit $2/3$ as $N \to \infty$.)

3. Suppose that for each integer $n \geq 2$ we have one square S_n of size $1/\sqrt{n} \times 1/\sqrt{n}$. Show that for any number $\epsilon > 0$, we can "tile" a 1×1 square with elements of the collection

$\{S_n\}$ such that the uncovered region of the 1×1 square has area less than ϵ. (No overlaps are allowed and all squares have to lie within the 1×1 square.)

Solution. Given a positive integer k, let R_k denote the k squares $S_{k^2}, S_{k^2+1}, \ldots, S_{k^2+k-1}$, arranged side-by-side in a row. The total width of R_k is

$$\frac{1}{\sqrt{k^2}} + \frac{1}{\sqrt{k^2+1}} + \cdots + \frac{1}{\sqrt{k^2+k-1}} < \frac{k}{\sqrt{k^2}} = 1,$$

and none of the squares in R_k has height greater than $1/k$. Thus, the row R_k fits within a rectangle of size $(1/k) \times 1$. Also, the total area of the squares in R_k is

$$\frac{1}{k^2} + \frac{1}{k^2+1} + \cdots + \frac{1}{k^2+k-1} > \frac{k}{k^2+k} = \frac{1}{k+1}.$$

so the area within the $(1/k) \times 1$ rectangle that is **not** covered by the squares in R_k is at most $1/k - 1/(k+1)$.

Since $k^2 + k - 1 < (k+1)^2$, the squares in R_k are different from those in any row $R_{k'}$ with $k' > k$. Thus, by stacking up rows $R_k, R_{k+1}, \ldots, R_{k+m}$ with m chosen so that

$$\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k+m} \le 1 < \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k+m+1},$$

we obtain an "almost-tiling" of the unit square into non-overlapping squares of the form S_n. The total area within the unit square that is **not** covered by the squares in this tiling is at most

$$\sum_{i=k}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) + \frac{1}{k+m},$$

where the i-th term in the sum accounts for the area not covered within the rectangle R_i and the last term accounts for a possible incomplete rectangular block at the top of the 1×1 square. The infinite series above telescopes and equals $1/k$, so the above bound is at most $1/k + 1/(k+m) < 2/k$. Since k was arbitrary, we can make this bound smaller than any given positive number ϵ. This proves the claim.

4. Let $a,\ b,\ c > 0$ and $abc = 1$. What are the largest and smallest possible values of

$$S = \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca}\ ?$$

Solution. The given expression S is always equal to 1. To see this, multiply the second term in S by a/a and the third term in S by ab/ab and simplify using $abc = 1$. This leads to

$$S = \frac{1}{1+a+ab} + \frac{a}{1+a+ab} + \frac{ab}{1+a+ab} = 1.$$

5. Let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha(1+x)}, \qquad 0 < \alpha < 1.$$

Find the choice of α that minimizes I_α. Explain.

Solution. We will show that the minimum occurs when $\alpha = 1/2$. Split I_α into $\int_0^1$ and $\int_1^\infty$. Setting $u = 1/x$, $du = -dx/x^2$ in the first integral leads to

$$\int_0^1 = \int_1^\infty \frac{du}{u^2 u^{-\alpha}(1+1/u)} = \int_1^\infty \frac{du}{u^{1-\alpha}(u+1)}.$$

Hence

$$I_\alpha = \int_0^1 + \int_1^\infty = \int_1^\infty (x^{-\alpha} + x^{\alpha-1}) \frac{dx}{x+1}.$$

To show that I_α is minimal at $\alpha = 1/2$, it suffices to show that for each fixed $x > 1$, the integrand above attains a minimum at $\alpha = 1/2$. Note that

$$\frac{\partial}{\partial \alpha}(x^{-\alpha} + x^{\alpha-1}) = (\log x)(-x^{-\alpha} + x^{\alpha-1}).$$

If $0 < \alpha < 1/2$, then $-\alpha > \alpha - 1$, so for $x > 1$, the above expression is positive. If $1/2 < \alpha < 1$, the inequality is reversed and the above expression is negative. Hence $(x^{-\alpha} + x^{\alpha-1})$ is decreasing in α for $0 < \alpha < 1/2$ and increasing in α for $1/2 < \alpha < 1$. It therefore must be minimal at $\alpha = 1/2$, as claimed.