UIUC Department of Mathematics

Mock Putnam Exam 1 Solutions

September 27, 1999

Problem 1. Show that $\sqrt{17-12\sqrt{2}}+\sqrt{17+12\sqrt{2}}$ is an integer.

Solution. Let x denote the given expression. Then

$$\begin{aligned} x^2 &= (17-12\sqrt{2}) + 2\sqrt{(17-12\sqrt{2})(17+12\sqrt{2})} + (17+12\sqrt{2}) \\ &= 34 + 2\sqrt{17^2 - 12^2\cdot 2} = 36, \end{aligned}$$

so $x = 6$, since x is positive.

Problem 2. Evaluate the infinite product

$$\left(1+2^{-1}\right)\left(1+2^{-2}\right)\left(1+2^{-4}\right)\left(1+2^{-8}\right)\dots$$

Solution. The infinite product is the limit of the partial products $P_n = \prod_{i=0}^{n}(1+2^{-2^i})$. We will show by induction that, for all $n \geq 0$, $(*)$ $P_n = 2(1-2^{-2^{n+1}})$. Thus, P_n tends to 2 as $n \to \infty$, and so the given infinite product is equal to 2. For $n = 0$, P_n reduces to $1+2^{-1}$, so $(*)$ holds in this case. Suppose now that $(*)$ holds for some $n \geq 0$. Then

$$P_{n+1} = P_n\left(1+2^{-2^{n+1}}\right) = 2\left(1-2^{-2^{n+1}}\right)\left(1+2^{-2^{n+1}}\right) = 2\left(1-2^{-2^{n+2}}\right),$$

so $(*)$ holds also for $n+1$. Hence, $(*)$ is true for all $n \geq 0$.

Remark. Alternatively, one could use induction to show that $P_n = \sum_{i=0}^{2^{n+1}-1} 2^{-i}$, which converges to $\sum_{i=0}^{\infty} 2^{-i} = 2$ when $n \to \infty$.

Problem 3. Show that there exists a multiple of 1999 that involves all 10 decimal digits.

Solution. Let N denote the number with decimal representation $123\dots900000$. Since successive multiples of 1999 differ by 1999, there exists such a multiple in the interval $[N, N+1999)$. Since every integer in that interval begins with the string $123\dots90$, that multiple has the desired property.

Remark. There are several other ways to prove the result. One approach starts with the observation that multiplication of 1999 by $1, 2, \dots 9$ generates numbers whose last digits are $1, 2, \dots, 9$, in some order. To get a single multiple involving all of these digits and 0, one simply uses a multiplier of the form $10\dots020\dots030\dots0\dots09$, where the digits $1, 2, \dots, 9$ are separated by sufficiently long blocks of 0's.

Problem 4. Let $b(n)$ denote the number of binary strings of length n which do not contain the substring 11. For example, $b(3) = 5$ since there are 5 such strings of length 3: 000, 001, 010, 100, 101. Express $b(n)$ in terms of familiar sequences.

Solution. We show that, for all $n \geq 1$, $(*)$ $b(n) = F_n$, where F_n is the n-th Fibonacci number, defined by $F_0 = F_1 = 1$, $F_2 = 3$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Since $b(1) = 1$, $b(2) = 3$ (the allowed strings are 00, 01, and 10), $(*)$ holds for $n = 1$ and $n = 2$. To prove that $(*)$ holds in general, it therefore suffices to show that $b(n)$ satisfies the same recurrence as the Fibonacci sequence. To see this, let $n \geq 2$ be given and split the admissible strings of length n (i.e., those counted by $b(n)$), into two groups: those ending with a 0, and those ending with a 1. The number of strings of the first type is $b(n-1)$ since any admissible string of length $n-1$ generates an admissible string of length n ending with 0 by tacking on a 0 at the end, and conversely any such string of length n arises in this way. The number of strings of the second type is $b(n-2)$, since if an admissible string ends in a 1, the second-last digit must be a 0, so the number of such strings of length n is equal to the number of admissible strings of length $n-1$ ending in a zero, which is the same as the total number of admissible strings of length $n-2$. Thus, $b(n) = b(n-1) + b(n-2)$ and so $b(n)$ satisfies the Fibonacci recurrence.

Problem 5. (Putnam, 1964) Let S be a set of $n \geq 1$ elements, and suppose $A_1, A_2, \dots A_k$ are distinct subsets of S such that (i) any two of these subsets intersect, and (ii) any subset of S that intersects each of the sets $A_1, A_2, \dots A_k$ must be one of the sets A_i. Show that $k = 2^{n-1}$.

Solution. Since the total number of subsets of S is 2^n, we need to show that the sets A_i comprise exactly half of all subsets. To this end, it is enough to show that, for each subset $A \subset S$, exactly one of the sets A and A^c is an A_i. Let $A \subset S$ be given. If A intersects A_i for all i, then $A = A_j$ for some j by property (i). In that case, A^c cannot be among the sets A_i, since A^c does not intersect $A_j = A$. If, on the other hand, A does not intersect a set A_j, then $A_j \subset A^c$. By property (i) this implies that A^c intersects each of the sets A_i, and by property (ii), it follows that $A^c = A_i$ for some i. As before, we see that in this case, A itself cannot be equal to an A_i. Hence the above claim holds.