Hints and Answers to Mock Putnam Exam 1

October 21, 1996

1. Write $a/c = d/b$ as a reduced fraction x/y. Then $a = hx$, $c = hy$, $d = kx$, $b = ky$ with positive integers h and k. Substituting this into $a^2 + b^2 + c^2 + d^2$, one obtains an expression which factors into $(h^2 + k^2)(x^2 + y^2)$ and thus is composite (since each factor is an integer ≥ 2).

2. Without loss of generality, let $n \geq m \geq 2$. The result follows from the two inequalities (1) $m^{-1/n} \geq 1 - \frac{\log m}{n}$ and (2) $n^{-1/m} \geq \frac{\log m}{n}$. To prove (1), use the fact that $e^{-x} > 1 - x$ for $x > 0$; to prove (2) it is enough to show that $n^{1/2} > \log n$ (since $n^{-1/m} \geq n^{1/2}/n$ and $\log n \geq \log m$), and this can be done, e.g., by showing that the function $f(x) = e^{x/2} - x$ is non-decreasing for $x \geq \log 2$ and positive at $x = \log 2$.

3. To find the limit $L = L(u)$ assuming its existence, just substitute L for u_n and u_{n-1} in the equation and solve for L. The resulting quadratic equation has two solutions, one positive and one negative, but the recurrence for u_n forces each term u_n, and hence L, to be positive. This gives $L = (u + \sqrt{u^2 + 4})/2$. To prove the existence of the limit, let $d_n = u_n - L$. The recurrence for u_n and the equation defining L gives $d_n = -d_{n-1}/(Lu_{n-1})$, and iterating this recurrence once leads to $d_n = d_{n-2}/(L^2 u_{n-1} u_{n-2})$. Since $L > 1$ and $u_{n-1}u_{n-2} > 1$ (which follows from the recurrence for u_{n-1}), this implies that d_n tends to 0 at a geometric rate and thus $u_n \to L$ as $n \to \infty$.

4. Writing $1/((2n-1)(2n+1)$ as $(1/2)(1/(2n-1) - 1/(2n+2))$, one sees that the series telescopes with sum $1/2$.

5. The answer is $P_n(n+1) = 2^{n+1} - 1$. To obtain this, define $Q_n(x) = P_n(x+1) - P_n(x)$. Then $Q_n(x)$ is a polynomial of degree $n-1$, and $Q_n(k) = 2^{k+1} - 2^k = 2^k = P_{n-1}(k)$ for $k = 0, 1, \ldots, n-1$. Since the polynomials $Q_n(x) = P_n(x+1) - P_n(x)$ and P_{n-1} have degree $n-1$ and agree on n points, they must be equal, i.e., $P_n(x+1) - P_n(x) = P_{n-1}(x)$ for all x. Setting $x = n$ gives $P_n(n+1) = P_{n-1}(n) + P_n(n) = P_{n-1}(n) + 2^n$. The claim $P_n(n+1) = 2^{n+1} - 1$ follows then easily by induction.

6. (Solution of Michael Roman) We have $0 = 0^2$, $1 = 1^2$, $2 = -1^2 - 2^2 - 3^2 + 4^2$, $3 = -1^2 + 2^2$, and $4 = k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2$ for $k = 0, 1, \ldots$. For $n = 4l + r$ with $l \geq 1$ and $0 \leq r \leq 3$ first represent r as above, then write $4l$ as a sum of l 4's and express each 4 as above using $k = 5$, $k = 9$, $k = 13$, etc., for the first square, thus ensuring that all squares occurring in the representation are different.