# UIUC Department of Mathematics

# Mock Putnam Exam 2 Solutions

# October 11, 1999

**Problem 1.** Let $x_1, x_2, \ldots, x_n$ be the numbers $1, 2, \ldots, n$, written in some order. Prove that

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \le 1^2 + 2^2 + \cdots + n^2.$$

**Solution.** Apply the Cauchy-Schwarz inequality:

$$\begin{aligned} &x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \\ &\quad \le \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} \left(x_2^2 + x_3^2 + \cdots + x_n^2 + x_1^2\right)^{1/2} \\ &\quad = x_1^2 + \cdots + x_n^2 = 1^2 + \cdots + n^2. \end{aligned}$$

An alternative solution is based on the inequality $ab \le (a^2 + b^2)/2$, which holds since $0 \le (a-b)^2 = a^2 + b^2 - 2ab$ for all real $a$, $b$. Applying this inequality to each of the products $x_1x_2, \ldots, x_nx_1$ and adding these inequalities gives again the result.

**Problem 2.** (Putnam 1990) Prove or disprove that $\sqrt{2}$ is the limit of a sequence of numbers of the form $n^{1/3} - m^{1/3}$, where $n, m = 0, 1, 2, \ldots$

**Solution.** Let $a_n = n^{1/3}$. The sequence $\{a_n\}$ is obviously increasing and tends to infinity. Moreover, since $a_{n+1} - a_n = (1/3) \int_n^{n+1} x^{-2/3} dx \le (1/3) n^{-2/3}$, the gap between consecutive terms of this sequence tends to zero. To prove the desired result, we will show, more generally, that any sequence $\{a_n\}$ of positive real numbers with these three properties (namely (i) it is increasing, (ii) it tends to infinity, and (iii) its gaps tend to zero) has the property that any given positive real number $\alpha$ is the limit of terms of the form $a_m - a_n$. To show this, assume a sequence $\{a_n\}$ and a number $\alpha$ with the stated properties are given, and fix $\epsilon > 0$; we need to show that there exist $n$, $m$ such that $a_n - a_m$ is within $\epsilon$ of $\alpha$. By properties (i) and (iii), we can choose $N = N(\epsilon)$ such that for all $n \ge N$, $0 < a_{n+1} - a_n < \epsilon$. It follows that the numbers $a_{N+k} - a_N$, $k = 0, 1, 2, \ldots$ form an increasing sequence with gaps at most $\epsilon$, that begins with 0 and, by property (ii), tends to infinity as $k \to \infty$. Hence there exists a term $a_{N+k} - a_N$ in this sequence that is within $\epsilon$ of $\alpha$, as wanted.

**Problem 3.** Prove that in the decimal expansion of $\sqrt{2}$ there is at least one non-zero digit between the millionth and the three-millionth decimal digits (inclusive) after the decimal point.

**Solution.** We will prove the claim by contradiction. Thus, assume that all digits between the millionth and three-millionth are zero. Let $r$ denote the rational number obtained by truncating $\sqrt{2}$ after the first million decimal digits, and define $\theta$ by $(*)$ $\sqrt{2} = r + \theta$. Note that (1) $r < 2$, and that $Nr$ is an integer, where $N = 10^{1,000,000}$, Also, since $\sqrt{2}$ is irrational, we have $\theta > 0$ and the assumption that all digits from the millionth to the three-millionth digits are zero implies (2) $0 < \theta < 10^{-3,000,000} = N^{-3}$. Multiplying $(*)$ by $N$, and squaring we get $2N^2 = (rN + \theta N)^2 = (rN)^2 + 2rN^2\theta + (\theta N)^2$. Since $2N^2$ and

$(rN)^2$ are integers, so must be the number $2rN^2\theta + (\theta N)^2$. But this is impossible, since by (1) and (2) $0 < 2rN^2\theta + (\theta N)^2 < 4N^{-1} + N^{-4} < 1$.

**Problem 4.** Let $s(n)$ denote the number of ways to write $n$ as a sum of positive odd integers, with the order taken into account. For example, $s(5) = 5$ since 5 can be represented as $1+1+1+1+1$, $1+1+3$, $1+3+1$, $3+1+1$, and 5. Find a formula for $s(n)$.

**Solution.** It is easy to check that $s(1) = 1$, $s(2) = 1$, $s(3) = 2$, $s(4) = 3$, and $s(5) = 5$. This suggests that, in general, $s(n) = F_{n-1}$, where $F_n$ is the $n$-th Fibonacci number. To show that this is indeed the case, it suffices to show that $s(n)$ satisfies the Fibonacci recurrence i.e., that $(*)$ $s(n) = s(n-1) + s(n-2)$ for all $n \geq 3$.
To obtain $(*)$, split the set of all representations of $n$ of the required form (i.e., as a sum of odd positive integers) into two classes: those in which the first term is 1, and those in which the first term is at least 3. Dropping the first term 1 in a representation of the first type, we see that the representations of this type are in 1-1 correspondence with the $s(n-1)$ representations of $n-1$. Moreover, by subtracting 2 from the first term of a representation of the second type, we see that those representations are in 1-1 correspondence with the $s(n-2)$ representations of $n-2$. Hence the total number of representations of $n$ is $s(n-1) + s(n-2)$, so $(*)$ holds.
**Alternative Solution.** One can also solve this problem via generating functions. The generating function for $s(n)$ is

$$\sum_{n=1}^{\infty} s(n)x^n = (x + x^3 + \cdots) + (x + x^3 + \cdots)^2 + \cdots$$

$$= \frac{x}{1-x^2} + \left(\frac{x}{1-x^2}\right)^2 + \cdots = \frac{x}{1-x-x^2},$$

which is $x$ times the generating function for Fibonacci numbers, $\sum_{n=0}^{\infty} F_n x^n$. Hence $s(n)$ must be equal to $F_{n-1}$.

**Problem 5.** Suppose 1000 points in the plane are given. Prove that there exists a circle such that there are exactly 500 points in the interior of the circle, and no point on the circle itself.

**Solution.** Given a point $P$ in the plane and a positive number $r$, let $C(P, r)$ denote the circle of radius $r$ around $P$. Clearly, for any choice of $P$, the number of points (among the 1000 given points) contained in the interior of $C(P, r)$ is 1000 if $r$ is sufficiently large, and 0 or 1 (depending on whether or not $P$ is one of the given points) if $r$ is sufficiently small. Furthermore, as $r$ increases, the number of points contained inside $C(P, r)$ increases in units of 1, provided $(*)$ *there is no value of $r$ such that there are two or more of the given points on the circle $C(P, r)$* .
If condition $(*)$ is satisfied, there must be a value of $r$ for which $C(P, r)$ has the required property. Thus, it remains to show that we can choose $P$ so that $(*)$ holds. To see this, note that for any two of the 1000 given points, the "bad" circles $C(P, r)$ that contain these two points must all have their center $P$ on a line perpendicular to the line segment joining these two points and going through the midpoint of that line segment. Since there are only finitely many pairs of two points, the bad locations for $P$ all fall onto one of finitely many lines. Hence, $(*)$ holds for all points $P$ that do not lie on any one of these lines.