UIUC Mock Putnam Exam 3/2000

November 6, 2000

Elementary Problems

E1 Suppose a and b are non-zero real numbers with $a + b = 1/a + 1/b$. Show that $a^3 + b^3 = 1/a^3 + 1/b^3$.

E2 Show that, for all positive integers n, the number $3^{n+2} + 4^{2n+1}$ is divisible by 13.

E3 What is the 2000th digit in the sequence $12345678910111213\dots$?

E4 How large can the product of a set of positive integers be if their sum is equal to 2000?

E5 In a round-robin tournament with n players, $P_1, P_2, \dots, P_n$, each of the players plays a match against every other player. There are no ties, so each match ends in a win for one side and a loss for the other side. Let W_k denote the number of wins of player P_k, and let L_k denote the number of losses of P_k. Show that $\sum_{k=1}^{n} W_k^2 = \sum_{k=1}^{n} L_k^2$.

Advanced Problems

A1 Show that, among any 10 consecutive integers, there is always one that has no common prime factor with any of the other integers.

A2 Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$.

A3 Show that, if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2.

A4 (Corrected version) Evaluate the infinite series $\sum_{n=-\infty}^{\infty} (-1)^n x^{n(n+1)/2}$ for $|x| < 1$.

A5 Determine, with proof, all functions f defined on the set of integers and satisfying $f(n + m) + f(n - m) = 2(f(m) + f(n))$ for all n and m.