

Hints and Answers to Mock Putnam Exam 3

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1. The three integers must be of the form $n_i = 2^{a_i}5^{b_i}$ ($i = 1, 2, 3$) with $a_i, b_i \geq 0$ and $\sum_{i=1}^3 a_i = 5$, $\sum_{i=1}^3 b_i = 5$. The problem then amounts to counting the number of such a_i and b_i . If different orderings of the three factors were counted separately, this number would be $\binom{5+2}{2} \binom{5+2}{2}$, the number of choices for (a_1, a_2, a_3) times the number of choices for (b_1, b_2, b_3) . The given problem is trickier as one has to avoid overcounting cases that differ only in the order of the factors; this can be achieved by only counting those triples $((a_1, b_1), (a_2, b_2), (a_3, b_3))$ that are in lexicographic order, i.e., where $a_1 \leq a_2 \leq a_3$ and if $a_i = a_{i+1}$ for some i , then $b_i < b_{i+1}$. The answer is $2 \cdot 21 + 3 \cdot 12 = 78$; $2 \cdot 21$ is the number of those cases where the a_i are distinct and $3 \cdot 12$ the number of cases where two of the a_i 's are equal.
2. The answer is $n(n+1)/2 + 1$; the proof is an easy induction.
3. First observe that, for each $m \geq 1$, there is exactly one power of two with m digits and 1 as leading digit, namely the first power of 2 that has m digits. It follows that, if $s(n)$ denotes the number of digits of n , then then $f(n) = s(2^n)$. One can easily show that $\lim_{n \rightarrow \infty} s(n)/\log_{10} n = 1$. Thus, $\lim_{n \rightarrow \infty} f(n)/n = \lim_{n \rightarrow \infty} s(2^n)/\log_2 2^n = \log_{10} 2$.
4. For $n = 1$, $n = 4$ and $n = 9$ the decomposition is obvious. A construction similar to that for $n = 6$ – a big square with side length $1 - 1/k$ and $2k - 1$ small squares with side length $1/k$ – gives a decomposition into $n = 2k$ squares for $k \geq 2$. This takes care of all even $n \geq 4$. To get odd values of n , say $n = 2k + 1$, simply take a decomposition for $n = 2k - 2$ and split one of the squares into four identical squares.
5. This was a recent Putnam problem. Assuming that no two points of distance 1 have the same color, consider two adjacent equilateral triangles of sidelength 1. The two vertices farthest apart in this configuration have distance $\sqrt{3}$ and must have identical colors. In fact, the same reasoning shows that any two points of distance $\sqrt{3}$ must have the same color. This leads to a contradiction by considering the points on a circle of radius $\sqrt{3}$.
6. Denote the cities by c_1, c_2, \dots, c_n , and for each c_i let A_i denote those cities c_j for which the (one way) road between c_i and c_j is from c_j to c_i , and B_i those where the road goes in the other direction. For the induction step from $n - 1$ to n , pick a city c_i for which $|A_i|$ is minimal; without loss of generality, assume c_n has this property. The induction hypothesis, applied to the cities c_1, \dots, c_{n-1} , gives the existence of a “capital” c_i for these $n - 1$ cities. Using the minimality of $|A_n|$ one can show that there is either a direct road from c_n to c_i or a road from c_n to one of the cities in A_i (from which there is, by definition, a road to c_i). Thus, c_i is in fact a capital for all n cities.