Math 213 Exam 1 (Solutions)

Prof. I.Kapovich March 6, 2007

Problem 1. Using induction, prove that for every integer $n \geq 1$ we have:

$$(\dagger) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

Solution.

We will prove that inequality $(\dagger)$ holds for every $n \geq 1$ by induction on n.

(1) For $n = 1$ the left hand side of $(\dagger)$ is $\frac{1}{1^2} = 1$ and the right-hand side of $(\dagger)$ is $2 - \frac{1}{1} = 2 - 1 = 1$. Since $1 \leq 1$, the statement has been verified for $n = 1$.

(2) Suppose now that $n \geq 1$ and that $(\dagger)$ has been established for n. We need to show that it holds for $n + 1$, that is we need to prove

$$(*) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \ldots \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$$

Since we know that for n inequality $(\dagger)$ holds, we add $\frac{1}{(n+1)^2}$ to both sides of $(\dagger)$ and get

$$(**) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \ldots \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

If we can now show that

$$(***) \qquad 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1},$$

this will imply that inequality $(*)$ holds as required.

We have:

$$\begin{aligned}
2-\frac{1}{n}+\frac{1}{(n+1)^2} \le 2-\frac{1}{n+1} &\iff \\
-\frac{1}{n}+\frac{1}{(n+1)^2} \le -\frac{1}{n+1} &\iff \\
\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1} \ge 0 &\iff \\
\frac{(n+1)^2-n-n(n+1)}{n(n+1)^2} \ge 0 &\iff \\
\frac{n^2+2n+1-n-n^2-n}{n(n+1)^2} \ge 0 &\iff \\
\frac{1}{n(n+1)^2} \ge 0 &
\end{aligned}$$

The last inequality in this sequence is obviously true since $n \ge 1$.

Thus $(***)$ holds and therefore $(*)$ also holds, which completes the inductive step.

Problem 2.

How many different strings can be made by permuting the letters of the word *ILLINOIS* if the two L's must be consecutive?

Solution.

We will apply the formula for the number of permutations with indistinguishable objects. Since the two L's must be consecutive, we consider them to be a single letter LL. Then the word $ILLINOIS$ has $n = 7$ letters: 3 I's, 1 LL, 1 N, 1 O and 1 S.

Hence the numbers of ways to permute letters of the word *ILLINOIS* if the two L's must be consecutive is:

$$\frac{7!}{3!\ 1!\ 1!\ 1!\ 1!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

Problem 3.

(a) Find the precise value of $\sum_{i=0}^{100} C(100, i)(-2)^i$.

(b) Find the coefficient at x^6 in the expansion of $(3 + x)^{75}$. Simplify the answer to the extent possible.

Solution.

(a) By the Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} C(n,i)a^{n-i}b^i.$$

Applying this formula with $a = 1$, $b = -2$, $n = 100$, we get

$$\sum_{i=0}^{100} C(100,i)(-2)^i = (1-2)^{100} = (-1)^{100} = 1.$$

(b) Apply the Binomial Theorem, stated above, with with $a = 3$, $b = x$, $n = 75$ and $i = 6$. Then the coefficient at x^6 is

$$C(75,6)3^{75-6} = \frac{75 \cdot 74 \cdot 73 \cdot 72 \cdot 71 \cdot 70}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 3^{69}.$$

Problem 4. A 4-card hand is chosen at random from a 52-card deck. Find the probability that there are at least 3 cards of the same suit in the hand.

Solution.

A 4-card hand with at least three cards of the same suit has either 3 cards of the same suit and one card of a different suit or it has all 4 cards of the same suit.

The number of 4-card hands with exactly 3 cards of the same suit and the 4-th card of a different suit is

$$C(4,1)C(13,3)C(39,1) = 4 \cdot \frac{13 \cdot 12 \cdot 11}{2 \cdot 3} \cdot 39 = 4 \cdot 13 \cdot 2 \cdot 11 \cdot 39 = 44616,$$

since there are $C(4,1)$ ways to choose a particular suit, $C(13,3)$ ways to choose a 3-card hand from that suit and $C(39,1)$ ways to choose the 4-th card from the remaining $52 - 13 = 39$ cards from the other suits.

The number of 4-card hands with all 4 cards of the same suit is

$$C(4,1)C(13,4) = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4} = 13 \cdot 2 \cdot 11 \cdot 10 = 2860.$$

Hence the probability of choosing a 4-card hand with at least 3 cards of the same suit is

$$\frac{56056 + 44616}{C(52,4)} = 47476 \cdot \frac{2 \cdot 3 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49} = 0.175366$$

Note.

A fairly common mistake in this problem was to say that the number of 4-card hands with at least 3 cards of the same suit equals

$$(\ddagger) \qquad C(4,1)C(13,3)C(49,1)$$

where there are $C(4,1)$ to choose a suit, $C(13,3)$ ways to pick a 3-card hand from that suit and $C(49,1)$ ways to "pick the fourth card out of the remaining $52-3=49$ cards". The problem with this argument is that every 4-card hand where all 4 cards are of the same suit is counted in formula (‡) more than once. Specifically, each such hand is counted in the above formula 4 times corresponding to 4 possible ways of splitting this hand into a 3-card hand and a 1-card hand. Observe, every hand with 3 cards of the same suit and the fourth card of a different suit is counted in (‡) once. The overall count in (‡), however, is incorrect.

Problem 5.

How many integers n, where $1 \leq n \leq 1000$, are divisible by 4 or 13?

Solution.

Let A be the set of all integers n, where $1 \leq n \leq 1000$, that are divisible by 4 and let B be the set of all integers n, where $1 \leq n \leq 1000$, that are divisible by 13. Then $A \cap B$ is the set of all in integers n, where $1 \leq n \leq 1000$, that are divisible by $4 \cdot 13 = 52$. We need to find $|A \cup B|$.

Since $1000 = 4 \cdot 250$, $1000 = 13 \cdot 76.9230$ and $1000 = 52 \cdot 19.23$, we have $|A| = 250$, $|B| = 76$ and $|A \cap B| = 19$. Therefore

$$|A \cup B| = |A| + |B| - |A \cap B| = 250 + 76 - 19 = 307.$$