Math 213 Exam 1 (Solutions)

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Problem 1[20 points]

For each of the following statements indicate whether it is true or false. You do not have to explain your answers.

(1) For the set $S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ we have $|S| = 1$.
(2) If $|A| = m$ and $|B| = n$, then the number of all functions from A to B is m^n.
(3) For every $n \geq r \geq 1$ we have $C(n,r) \leq P(n,r)$.
(4) Whenever E, F are events such that $p(E) = 0$, then E and F are independent.
(5) For every $n \geq 1$ the number $\sum_{i=0}^{n} \binom{n}{i}$ is even.

Solution.

(1) FALSE. In fact, we have $|S| = 3$.
(2) FALSE. In fact, he number of all functions from A to B is n^m.
(3) TRUE. Indeed, $C(n,r) = P(n,r)/r!$ and so $C(n,r) \leq P(n,r)$.
(4) TRUE. Indeed, in this case $p(E) = 0$ and $0 \leq p(E \cap F) \leq p(E) = 0$, so that $p(E \cap F) = 0$. Thus $0 = 0 \cdot p(F)$, that is $p(E \cap F) = p(E)p(F)$.
(5) TRUE. Indeed, $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ is even.

Problem 2[20 points] Prove that for every integer $n \geq 1$ we have:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}.$$

Give all the details of your work.

Solution.

We will prove this statement by induction.

Base of Induction.

For $n = 1$ we have $1^2 = 1 = \frac{1(1+1)}{2}$, as required.

Inductive Step.

Suppose that $n \geq 1$ and that it is known that

$$(*) \qquad 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

We need to show that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 + (-1)^n (n+1)^2 = (-1)^n \frac{(n+1)(n+2)}{2}.$$

By adding $(-1)^n (n+1)^2$ to both sides of $(*)$ we get:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 + (-1)^n (n+1)^2 = (-1)^{n-1} \frac{n(n+1)}{2} + (-1)^n (n+1)^2 =$$

$$(-1)^n (n+1) \left(-\frac{n}{2} + (n+1) \right) = (-1)^n (n+1) \frac{2n+2-n}{2} = (-1)^n \frac{(n+1)(n+2)}{2},$$

as required.

Problem 3[20 points]

A basket contains oranges, apples, pears and grapefruits. How many ways of choosing six pieces of fruit from the basket are there, so that exactly two of the pieces chosen are oranges?

Provide a detailed explanation of your answer.

Solution To choose six pieces of fruit containing exactly two oranges we need to pick two oranges from the basket (there is one way of doing that), put them aside and then pick four pieces of fruit out of apples, pears and grapefruits.

Therefore the number of ways to choose six pieces of fruit from the basket are there, so that exactly two of the pieces chosen are oranges, is equal to the number of 4-combinations with repetitions out of a set with 3 elements. This number is equal to:

$$C(4+3-1, 3-1) = C(6,2) = \frac{6 \cdot 5}{2} = 15.$$

Problem 4[20 points]

Find the precise value of $\sum_{i=1}^{100} C(100,i)(-3)^i$. Give all the details of your work.

Solution.

By the Binomial Theorem we have $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$.

Putting $n = 100$, $x = 1$, $y = -3$, we get

$$(1-3)^{100} = \sum_{i=0}^{100} \binom{n}{i} 1^{n-i}(-3)^i = \sum_{i=0}^{100} \binom{n}{i} (-3)^i =$$

$$= \binom{n}{0} 1^{100}(-3)^0 + \sum_{i=1}^{100} C(100,i)(-3)^i = 1 + \sum_{i=1}^{100} C(100,i)(-3)^i.$$

Therefore

$$\sum_{i=1}^{100} C(100,i)(-3)^i = (1-3)^{100} - 1 = 2^{100} - 1.$$

Problem 5[20 points] A 3-card hand is chosen at random from a 52-card deck. Find the probability that there are at least 2 spades in the hand.

Give all the details of your work and simplify the answer to the extent possible.

Solution.

The number of all 3-card hands is $C(52,3) = \frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3} = 22,100$.

If a 3-card hand contains at least 2 spades, then it has either 2 or 3 spades. There are 13 spades and $39 = 52 - 13$ non-spades in the deck.

Therefore the number of 3-card hands containing exactly 2 spades is

$$C(13,2)C(39,1) = \frac{13 \cdot 12}{2} \cdot 39 = 3,042.$$

Similarly, the number of 3-card hands containing exactly 3 spades is

$$C(13,3) = \frac{13 \cdot 12 \cdot 11}{2 \cdot 3} = 286.$$

Therefore the probability that a 3-card hand out of the 52-card deck contains at least two spades is

$$\frac{C(13,2)C(39,1) + C(13,3)}{C(52,3)} = \frac{3,042 + 286}{22,100} = \frac{3328}{22,100} \approx 0.1506$$