

Math 213, Quiz 6 (Solutions); Friday, February 29, 2008

1.

Let $n \geq 2$ be an integer.

A bit-string of length n is chosen at random. Let $X = 3j$ where j is the number of 1-bits in the string chosen. Find EX and VX .

Provide a detailed explanation of your answer.

Solution.

For $i = 1, 2, \dots, n$ let

$$X_i = \begin{cases} 3, & \text{if } i\text{-th bit is 1} \\ 0, & \text{if } i\text{-th bit is 0.} \end{cases}$$

Then $X = X_1 + \dots + X_n$ and hence $EX = EX_1 + \dots + EX_n$. Moreover, since X_1, \dots, X_n are clearly independent, we also have $VX = VX_1 + \dots + VX_n$.

We have

$$EX_i = 3 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{3}{2},$$

$$E(X_i^2) = 3^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{2} = \frac{9}{2},$$

and hence

$$VX_i = E(X_i^2) - (EX_i)^2 = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}.$$

Therefore

$$EX = EX_1 + \dots + EX_n = \frac{3n}{2}$$

and

$$VX = VX_1 + \dots + VX_n = \frac{9n}{4}$$

Alternative Solution.

Let $Y = j$ be the number of 1-bits in the string chosen. Then $X = 3Y$ and the random variable Y has the binomial distribution with the parameters $p = q = 1/2$ on the set $\{0, 1, \dots, n\}$. Therefore, by the results of Example 5 and Example 18 in Ch 6.4 we have $EX = np = n/2$ and $VX = npq = n/4$.

Since $X = 3Y$, we have $EX = 3EY = 3n/2$ and $VX = 3^2VY = 9n/4$.