Math 280 Section C1 Exam 1 (WITH SOLUTIONS)
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Problem 1.[20 points] Select the correct answer for each of the following questions. (Each question has exactly one correct answer) In this problem you do not have to show work, you just need to indicate the correct answers!

(1) Consider the function

$$f(x,y) = \begin{cases} x^2 + y^2, \text{ if } (x,y) \neq (0,0) \\ 1, \text{ if } (x,y) = (0,0) \end{cases}$$

Then:

(a) The function $f(x,y)$ is continuous at $(0,0)$.
(b) The limit
$$\lim_{x \to 0, y \to 0} f(x,y)$$
does not exist.
(c) The limit
$$\lim_{x \to 0, y \to 0} f(x,y)$$
exists
(d) None of the above.

Answer: C

(2) Let $f(x,y)$ be a function such that $\frac{\partial f}{\partial x}(1,1) = 3$ and $\frac{\partial f}{\partial y}(1,1) = 5$.
Then:

(a) The function f is differentiable at $(1,1)$
(b) The function f is not differentiable at $(1,1)$.
(c) $df|_{(1,1)} = 3dx + 5dy$.
(d) The function f is not guaranteed to be differentiable at $(1,1)$.

Answer: D

(3) Let $f(x,y)$ be a real-valued function defined in $\mathbb{E}^2$. Suppose $f(x,y)$ is continuous and differentiable in $\mathbb{E}^2$ and suppose $f(x,y)$ has exactly two critical points: $(1,3)$ and $(2,-9)$.
Then:

(a) The funcvtion f has at least one relative maximum.
(b) The function f has at least one relative minimum.
(c) $10\frac{\partial f}{\partial x}|_{1,3)} - 7\frac{\partial f}{\partial y}|_{(2,-9)} = 0$
(d) The function f has a saddle point.

Answer: C

(4) Let $f(x,y)$ be a real valued function $f : \mathbb{E}^2 \longrightarrow \mathbb{R}$ such that all partial derivatives of orders one and two exist and are continuous everywhere in $\mathbb{E}^2$.
Then

(a) $\frac{\partial^2 f}{\partial x \partial y}(0,2) - \frac{\partial^2 f}{\partial y \partial x}(0,2) \neq 0$.
(b) $\frac{\partial^2 f}{\partial x \partial y}(0,2) - \frac{\partial^2 f}{\partial y \partial x}(0,2) = 0$
(c) $\frac{\partial^3 f}{\partial x^3}(1,1)$ always exists.
(d) None of the above.

Answer: B

Problem 2.[20 points]
Let $F(u,v,x) = (F_1(u,v,x), F_2(u,v,x))$ be a function $F : \mathbb{E}^3 \longrightarrow \mathbb{E}^2$ such that at the point (u_0, v_0, x_0) the Jacobi matrix of F is:

$$F'|_{(u_0,v_0,x_0)} = \begin{bmatrix} 1 & 5 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

Suppose $F_1(u_0,v_0,x_0) = F_2(u_0,v_0,x_0) = 0$ and suppose that the system of equations

$$F_1(u,v,x) = 0, F_2(u,v,x) = 0$$

defines implicit functions $u = u(x)$ and $v = v(x)$ such that $u(x_0) = u_0$ and $v(x_0) = v_0$. Compute the derivatives

$$\frac{\partial u}{\partial x}(x_0) \text{ and } \frac{\partial v}{\partial x}(x_0)$$

Solution.
By the Implicit Function Theorem we have:

$$\frac{\partial u}{\partial x}|_{x_0} = -\frac{\frac{\partial(F_1,F_2)}{\partial(x,v)}}{\frac{\partial(F_1,F_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & 5 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix}} = -\frac{-5}{-16} = -\frac{5}{16}$$

and

$$\frac{\partial v}{\partial x}|_{x_0} = -\frac{\frac{\partial(F_1,F_2)}{\partial(u,x)}}{\frac{\partial(F_1,F_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix}} = -\frac{1}{-16} = \frac{1}{16}.$$

Problem 3.[20 points] Find all the relative extrema (if it has any) of the function $f(x,y) = 2xy + 4y - x^2 - 3y^2 + 5$.
Solution.
We will first find the critical points.

$$\frac{\partial f}{\partial x} = 2y - 2x, \quad \frac{\partial f}{\partial y} = 2x + 4 - 6y$$
$$2y - 2x = 0, 2x + 4 - 6y = 0 \Longrightarrow x = y = 1.$$

Thus $f(x,y)$ has exactly one critical point: $(1,1)$. We will now apply the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = -2 = A, \quad \frac{\partial^2 f}{\partial y^2} = -6 = C, \quad \frac{\partial^2 f}{\partial y \partial x} = 2 = B.$$

Since $B^2 - AC = 4 - 12 = -8 < 0$ and $A + C = -8 < 0$, the point $(1,1)$ is a relative minimum.
Problem 4.[20 points] Let $F(x,y,z) = 3xy - x^3 + 5yz + z^2$. Find the directional derivative $\bigtriangledown_{\mathbf{v}} F|_{(0,1,-1)}$, where $\mathbf{v}$ is a normal vector to the surface

$$3x^2 - 2xyz + y^3 + z = 0$$

at the point $(0,1,-1)$ such that the first coordinate of $\mathbf{v}$ is positive.

Solution.

We will first find the vector $\mathbf{v}$. Denote $G(x,y,z) = 3x^2 - 2xyz + y^3 + z$. Since $\mathbf{v}$ is normal to the surface $G = 0$, then $\mathbf{v}$ is orthogonal to the gradient vector $\bigtriangledown G|_{(0,1,-1)}$.

We have

$$\bigtriangledown G = [6x - 2yz, -2xz + 3y^2, -2xy + 1]$$
$$\bigtriangledown G|_{(0,1,-1)} = [2,3,1].$$

Thus we can take $\mathbf{v} = [2,3,1]$. Then $||\mathbf{v}|| = \sqrt{4+9+1} = \sqrt{14}$. We also have

$$\bigtriangledown F|_{(0,1,-1)} = [3y - 3x^2, 3x + 5z, 5y + 2z]|_{(0,1,-1)} = [3,-5,3].$$

Therefore

$$\bigtriangledown_{\mathbf{v}} F|_{(0,1,-1)} = \bigtriangledown F|_{(0,1,-1)} \cdot \frac{\mathbf{v}}{||\mathbf{v}||} = (1/\sqrt{14})[3,-5,3] \cdot [2,3,1] = -\frac{6}{\sqrt{14}}.$$

Problem 5.[20 points]

Let $f(x,y)$ be a function differentiable at $(1,3)$ and such that $\frac{\partial f}{\partial x}(1,3) = -2$, $\frac{\partial f}{\partial y}(1,3) = 7$ and $f(1,3) = 10$.

Estimate the value $f(1.01, 2.98)$.

Solution

Since f is differentiable at $(x,y) = (1,3)$, we have

$$f(x + \triangle x, y + \triangle y) \approx f(x,y) + \frac{\partial f}{\partial x}\triangle x + \frac{\partial f}{\partial y}\triangle y$$

Hence

$$f(1.01, 2.98) \approx 10 - 2 \cdot 0.01 + 7 \cdot (-0.02) = 10 - 0.02 - 0.14 = 9.84$$