

Math 280 Section C1 Exam 2 (SOLUTIONS)

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**Problem 1.** Select the correct answer for each of the following questions. (Each question has exactly one correct answer) In this problem you do not have to show work, you just need to indicate the correct answers!

(1) Let  $f(x, y)$  be a function which is continuous in the entire  $xy$ -plane.

Then:

- (a) We have  $\text{curl grad } f = 0$  everywhere in the  $xy$ -plane.
- (b) For some point  $(x_0, y_0)$  we have  $(\text{curl grad } f)|_{(x_0, y_0)} \neq 0$ .
- (c) There is always a point  $(x_0, y_0)$  such that  $(\text{curl grad } f)|_{(x_0, y_0)}$  does not exist.
- (d) None of the above.

Answer: (d)

(2) Let  $f(x, y)$  be a continuous function in the entire  $xy$ -plane.

Then the integral  $\int_1^3 (\int_0^{2x-2} f(x, y) dy) dx$

- (a) is equal to  $\int_1^3 (\int_0^{2x-2} f(x, y) dx) dy$ ;
- (b) is equal to  $\int_0^4 (\int_3^{1+y/2} f(x, y) dx) dy$ ;
- (c) is equal to  $\int_0^4 (\int_{1+y/2}^3 f(x, y) dx) dy$ ;
- (d) does not always exist.

Answer: (c)

(3) Consider the vector-field  $\mathbf{u} = (2x + 3y - 2)\mathbf{i} + y^3\mathbf{j}$ . Suppose  $C$  is a smooth simple closed curve in the  $xy$ -plane traveled counter-clockwise and let  $\mathbf{T}$  denote the unit tangent vector to  $C$ . Suppose that the region  $R$  bounded by  $C$  has area 10.

Then the integral

$$\oint_C \mathbf{u}_T ds$$

- (a) is equal to  $-30$ ;
- (b) is equal to  $0$ ;
- (c) is equal to  $30$ ;
- (d) does not always exist.

Answer: (a)

(4) Let  $F(x, y)$  be a function which is continuous and has continuous first partial derivatives in a certain domain  $D$  containing the points  $(0, 0)$  and  $(1, 1)$ . Suppose  $F(1, 1) = 5$ ,  $F(0, 0) = 7$  and denote  $P(x, y) = \frac{\partial F}{\partial x}$ ,  $Q(x, y) = \frac{\partial F}{\partial y}$ .

Then

- (a) The integral  $\int P dx + Q dy$  is not necessarily independent of path in  $D$ .
- (b) For any smooth curve  $C$  from  $(0, 0)$  to  $(1, 1)$  in  $D$

$$\int_C^{(1,1)} 3P dx + 3Q dy = -6.$$

- (c) The integral  $\int P dx + Q dy$  is independent of path in the  $xy$ -plane.
- (d) None of the above.

Answer: (b)

**Problem 2.**[20 points]

Let

$$F(t) = \int_1^{t^2} \ln(x^2 + t^2) dx.$$

Compute in the explicit form (eliminating the integrals) the derivative  $\frac{d}{dt} F(t)$ .

**Solution.**

By the Leibnitz rule we have:

$$\begin{aligned} \frac{d}{dt} F(t) &= 2t \ln(t^4 + t^2) + \int_1^{t^2} \frac{2t}{x^2 + t^2} dx = \\ &= 2t \ln(t^4 + t^2) + 2t \int_1^{t^2} \frac{2t}{x^2 + t^2} dx = 2t \ln(t^4 + t^2) + \frac{2t}{t} [\arctan \frac{x}{t}]_{x=1}^{x=t^2} = \\ &= 2t \ln(t^4 + t^2) + 2 \arctan t - 2 \arctan \frac{1}{t}. \end{aligned}$$

**Problem 3.**[20 points] Compute the integral

$$\iint_{x^2+y^2 \leq 5} x^4 + 2x^2y^2 + y^4 dx dy$$

**Solution.**

We will perform a change of variables  $x = r \cos \theta$ ,  $y = r \sin \theta$ :

$$\begin{aligned} \iint_{x^2+y^2 \leq 5} x^4 + 2x^2y^2 + y^4 dx dy &= \iint_{x^2+y^2 \leq 5} (x^2 + y^2)^2 dx dy = \iint_{\substack{0 \leq r \leq \sqrt{5} \\ 0 \leq \theta \leq 2\pi}} r^4 r dr d\theta = \\ &= \int_0^{\sqrt{5}} \left( \int_0^{2\pi} r^5 d\theta \right) dr = 2\pi \int_0^{\sqrt{5}} r^5 dr = 2\pi \left[ \frac{r^6}{6} \right]_0^{\sqrt{5}} = \frac{125\pi}{3} \end{aligned}$$

**Problem 4.**[20 points]

Let  $C$  be the boundary of the rectangle with vertices  $(0, 0)$ ,  $(3, 0)$ ,  $(0, 5)$ ,  $(3, 5)$  traveled counter-clockwise.

Compute the integral

$$\oint_C (x^2y^2 + e^{x^2}) dx + (\cos(y^4 + 1) + x - 3) dy$$

**Solution.**

By the Green's theorem we have:

$$\begin{aligned} \oint_C (x^2y^2 + e^{x^2}) dx + (\cos(y^4 + 1) + x - 3) dy &= \iint_{\substack{0 \leq y \leq 5 \\ 0 \leq x \leq 3}} (1 - 2yx^2) dx dy = \int_0^3 \left( \int_0^5 (1 - 2yx^2) dy \right) dx = \\ &= \int_0^3 [y - y^2x^2]_{y=0}^{y=5} dx = \int_0^3 (5 - 25x^2) dx = [5x - 25x^3/3]_0^3 = \\ &= 15 - 25 \cdot 27/3 = -210. \end{aligned}$$

**Problem 5.**[20 points]

Let  $C$  be the graph of the function  $y = x^2 - 3$  for  $0 \leq x \leq 1$ . Compute the integral

$$\int_C^{(1,-2)}_{(0,-3)} x \, ds$$

**Solution.**

We will use the parameterization of  $C$  given by  $x = t, y = t^2 - 3$ , where  $0 \leq t \leq 1$ .

Then  $ds = \sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1 + 4t^2} dt$  and

$$\int_C^{(1,-2)}_{(0,-3)} x \, ds = \int_0^1 t \sqrt{1 + 4t^2} dt.$$

We will use the substitution  $u = 1 + 4t^2$ , so that  $du = 8t dt$  and  $t dt = (1/8)du$ . Then:

$$\int_0^1 t \sqrt{1 + 4t^2} dt = \frac{1}{8} \int_1^5 u^{1/2} du = \frac{1}{8} \left[ \frac{2}{3} u^{3/2} \right]_1^5 = \frac{5^{3/2} - 1}{12}.$$