# H/wk no 1

§2.4

2. Represent the following functions by drawing level curves.

(a) $z = 3 - x - 3y$

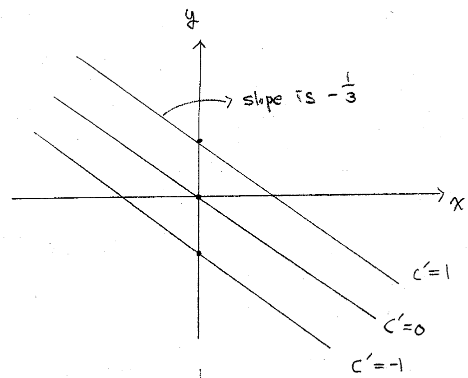
$\Leftrightarrow f(x, y) = 3 - x - 3y.$

If $f(x, y) = c$ then $f$ is called the level curve

So $3 - x - 3y = c$

Therefore $y = \frac{-x}{3} + \underbrace{\frac{1+c}{3}}_{\text{let } c'}$

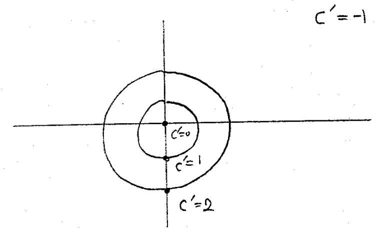
$\therefore y = -\frac{x}{3} + c'$




(b) $z = x^2 + y^2 + 1 \Leftrightarrow f(x, y) = x^2 + y^2 + 1$

If $f(x, y) = c$ then $f(x, y) = x^2 + y^2 + 1 = c$

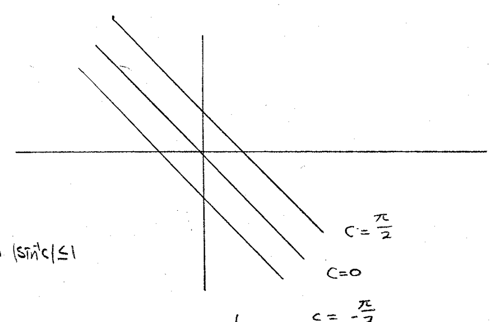
So $x^2 + y^2 + 1 = c \quad \therefore x^2 + y^2 = c'$




(c) $z = \sin(x+y) \Leftrightarrow f(x, y) = \sin(x+y)$

If $f(x, y) = c$ then $\sin(x+y) = c$

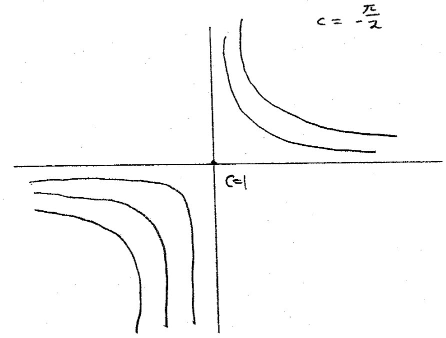
So $x + y = \sin^{-1} c \quad \therefore y = -x + \underbrace{\sin^{-1} c}_{\text{If } |c| \le \frac{\pi}{2} \text{ then } |\sin^{-1} c| \le 1}$




(d) $z = e^{xy} \Leftrightarrow f(x, y) = e^{xy}$

If $f(x, y) = c$ then $e^{xy} = c > 0$

So $xy = \ln c \quad \therefore y = \frac{\ln c}{x}$



4. Determine the values of the following limits, wherever the limit exists.

(a) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2-y^2}{1+x^2+y^2} = \dfrac{0}{1} = 0$

(b) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x}{x^2+y^2}$ = no limit $\because \lim\limits_{x\to 0} \dfrac{x}{x^2} = \dfrac{1}{0} = \infty$ $\ast$ $\lim\limits_{y\to 0} \dfrac{0}{y^2} = 0$

(c) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{(1+y^2)\sin x}{x} = \lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{\sin x}{x} = 1$

(d) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1+x-y}{x^2+y^2} = \dfrac{1}{0} = \infty$

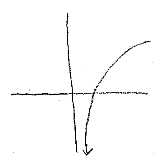
5. Show that the following functions are discontinuous at (0,0)

(a) $z = \dfrac{x}{x-y}$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{x-y} \quad \begin{cases} \text{if } y=0 \Rightarrow \lim\limits_{x\to 0} \dfrac{x}{x} = 1 \\ \text{if } x=0 \Rightarrow \lim\limits_{y\to 0} \dfrac{0}{y} = 0 \\ \text{if } x=y \Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x}{0} = \infty \end{cases} \ast \quad \therefore \text{ no limit}$$

(b) $z = \log(x^2+y^2)$

$$\begin{cases} \text{if } y=0 \Rightarrow \lim\limits_{x\to 0} \log x^2 \\ \text{if } x=0 \Rightarrow \lim\limits_{y\to 0} \log y^2 \\ \text{if } x=y \Rightarrow \lim\limits_{\substack{y\to 0\\ x\to 0}} \log(x^2+y^2). \end{cases}$$



7. Prove the theorem: Let $f(x,y)$ be defined in domain D and continuous at the point $(x_1, y_1)$ of D. If $f(x_1,y_1) > 0$, then there is a neighborhood of $(x_1,y_1)$ in which $f(x,y) > \frac{1}{2} f(x_1,y_1) > 0$ [Hint: Use $\varepsilon = \frac{1}{2} f(x_1,y_1)$ in the definition of conti

Pf) Since $f(x,y)$ is continuous at $(x_1,y_1)$, $\lim\limits_{\substack{x\to x_1\\ y\to y_1}} f(x,y) = f(x_1,y_1)$

This means that given $\varepsilon>0$, $\exists \delta>0$ s.t. $|f(x,y) - f(x_1,y_1)| < \varepsilon$ if $0 < (x-x_1)^2 + (y-y_1)^2 < \delta^2$

Choose $\varepsilon = \frac{1}{2} f(x_1, y_1)$

Then $|f(x, y) - f(x_1, y_1)| < \varepsilon = \frac{1}{2} f(x_1, y_1)$

So $-\frac{1}{2} f(x_1, y_1) < f(x, y) - f(x_1, y_1) < \frac{1}{2} f(x_1, y_1)$

$\therefore \quad 0 < \frac{1}{2} f(x_1, y_1) < f(x, y) < \frac{3}{2} f(x_1, y_1)$

§ 2.6

1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(a) $z = \dfrac{y}{x^2+y^2}$

(i) $\dfrac{\partial z}{\partial x} = \dfrac{-y(2x)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

(ii) $\dfrac{\partial z}{\partial y} = \dfrac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

(b) $z = y \sin xy$

(i) $\dfrac{\partial z}{\partial x} = 0 \cdot \sin xy + y[(\cos xy) \cdot y] = y^2 \cos xy$

(ii) $\dfrac{\partial z}{\partial y} = \sin xy + y[(\cos xy)x] = \sin xy + xy \cos xy$

(c) $x^3 + x^2y - x^2z + z^3 - 2 = 0$

(i) $3x^2 + 2xy - 2xz - x^2 \dfrac{\partial z}{\partial x} + 3z^2 \dfrac{\partial z}{\partial x} = 0$

$\Rightarrow 3x^2 + 2xy - 2xz = (x^2 - 3z^2) \dfrac{\partial z}{\partial x}$

$\Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{3x^2 + 2xy - 2xz}{x^2 - 3z^2}$

(ii) $x^2 - x^2 \dfrac{\partial z}{\partial y} + 3z^2 \dfrac{\partial z}{\partial y} = 0 \Rightarrow \dfrac{\partial z}{\partial y} = \dfrac{x^2}{x^2 - 3z^2}$

(d) $z = \sqrt{e^{x+2y} - y^2}$

(i) $\dfrac{\partial z}{\partial x} = \frac{1}{2}(e^{x+2y} - y^2)^{-\frac{1}{2}}(e^{x+2y}) = \dfrac{e^{x+2y}}{2\sqrt{e^{x+2y} - y^2}}$

(ii) $\dfrac{\partial z}{\partial y} = \frac{1}{2}(e^{x+2y} - y^2)^{-\frac{1}{2}}\{(2e^{x+2y}) - 2y\} = \dfrac{e^{x+2y} - y}{\sqrt{e^{x+2y} - y^2}}$

(e) $z = (x^2+y^2)^{3/2}$

(i) $\frac{\partial z}{\partial x} = \frac{3}{2}(x^2+y^2)^{1/2}(2x) = 3x\sqrt{x^2+y^2}$

(ii) $\frac{\partial z}{\partial y} = \frac{3}{2}(x^2+y^2)^{1/2}(2y) = 3y\sqrt{x^2+y^2}$

(f) $z = \arcsin(x+2y)$

(i) $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x+2y)^2}} \cdot 1$ <Note> $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-(x+2y)^2}} \cdot 2$

(g) $e^x + 2e^y - e^z - z = 0$

(i) $e^x - e^z\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 0 \Rightarrow e^x = (1+e^z)\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{e^x}{1+e^z}$

(ii) $2e^y - e^z\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 0 \Rightarrow 2e^y = (1+e^z)\frac{dz}{dy} \Rightarrow \frac{\partial z}{\partial y} = \frac{2e^y}{1+e^z}$

(h) $xy^2 + yz^2 + xyz = 1$

(i) $y^2 + 2yz\frac{\partial z}{\partial x} + yz + x(y)\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-y(y+z)}{y(x+2z)}$

$\therefore \frac{\partial z}{\partial x} = \frac{-(y+z)}{x+2z}$

(ii) $2xy + z^2 + 2yz\frac{\partial z}{\partial y} + xz + xy\frac{\partial z}{\partial y} = 0$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(2xy+z^2+xz)}{y(x+2z)}$

3. Evaluate the indicated partial derivatives.

(a) $u = x^2 - y^2$, $v = x - 2y$ (i) $\left(\frac{\partial u}{\partial x}\right)_y = 2x$ (ii) $\left(\frac{\partial v}{\partial y}\right)_x = -2$

(b) $x = e^u\cos v$, $y = e^u\sin v$ (i) $\left(\frac{\partial x}{\partial u}\right)_v = e^u\cos v$ · (ii) $\left(\frac{\partial y}{\partial v}\right)_u = e^u\cos v$

(c) $u = x - 2y$, $v = u - 2y$ $\Rightarrow$ $x = u + 2y$ $y = \frac{u-v}{2}$ (i) $\left(\frac{\partial x}{\partial u}\right)_y = 1$ (ii) $\left(\frac{\partial y}{\partial v}\right)_u = -\frac{1}{2}$

(d) $r = \sqrt{x^2+y^2}$, $x = r\cos\theta$ $\Rightarrow$ (i) $\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$ (ii) $\left(\frac{\partial r}{\partial \theta}\right)_x = x\sec\theta\tan\theta$