H/Wk 12

6. (a) $P = 2xy \Rightarrow \frac{\partial P}{\partial y} = 2x$

$Q = x^2 - y^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (ie independent of path)

$$\int_{(1,1)}^{(x,y)} 2xy\,dx + (x^2-y^2)\,dy = F(x,y) - F(1,1)$$

$$= x^2y - \frac{1}{3}y^3 - (1-\frac{1}{3})$$

$$= x^2y - \frac{y^3}{3} - \frac{2}{3}$$

(b) $P = \sin y \Rightarrow \frac{\partial P}{\partial y} = \cos y$

$Q = x\cos y \Rightarrow \frac{\partial Q}{\partial x} = \cos y$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial F}{\partial x} = -\sin y \Rightarrow F = x\sin y + C(y)$

$\frac{\partial F}{\partial y} = x\cos y + C'(y) = x\cos y$. So $C(y)=0$ $F = x\sin y$

$$\int_{(0,0)}^{(x,y)} \sin y\,dx + x\cos y\,dy = x\sin y - 0 = x\sin y$$

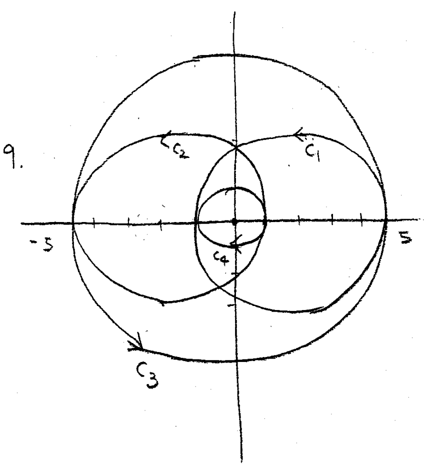
7. $$\oint \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2} = \oint_{x^2+y^2=1} \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2} = \int_0^{2\pi} [-\cos^2\theta \sin\theta \sin\theta - \cos^3\theta\cos\theta]\,d\theta$$

$x = \cos\theta$
$y = \sin\theta$ $0 \le \theta \le 2\pi$

$$= -\int_0^{2\pi} (\cos^2\theta \sin^2\theta + \cos^4\theta)\,d\theta$$

$$= -\left[\frac{\sin\theta\cos\theta}{2} + \frac{x}{2}\right]_0^{2\pi} = -\pi.$$

9.



$$\oint_{C_3} P\,dx + Q\,dy = \oint_{C_2} P\,dx + Q\,dy + \oint_{C_1} P\,dx + Q\,dy - \oint_{C_4} P\,dx + Q\,dy$$

$$\Rightarrow 13 = 9 + 11 - \oint_{C_4} P\,dx + Q\,dy$$

$$\therefore \oint_{C_4} P\,dx + Q\,dy = 7$$

10. (a) $F(x,y) = x^2 - y^2 \Rightarrow \nabla F = [2x, -2y]$

$$\int_{(0,0)}^{(2,8)} \nabla F \cdot d\vec{r} = F(2,8) - F(0,0) = -60$$

(b) $$\oint_C \frac{\partial F}{\partial n}\,ds = \oint_C (\nabla F \cdot \vec{n})\,ds = \iint_{x^2+y^2\le 1} \operatorname{div}(\nabla F)\,dx\,dy = \iint_{x^2+y^2\le 1} (2-2)\,dx\,dy = 0$$

Ch 5.10 1

(a) $$\int_{(1,0,0)}^{(1,0,2\pi)} z\,dx + x\,dy + y\,dz \;\overset{\begin{cases} x=\cos t \\ y=\sin t \\ z=t \end{cases}\; 0\le t\le 2\pi}{=\!=}\; \int_0^{2\pi} (t\sin t + \cos^2 t + \sin t)\,dt = 3\pi$$

(b) $$\int_{(1,0,1)}^{(2,3,2)} x^2\,dx - xz\,dy + y^2\,dz, \quad \vec{V} = [1, 3, 1] \quad \begin{cases} x = 1+t \\ y = 3t \\ z = 1+t \end{cases}$$

$\|$

$$\int_0^1 (1+t)^2\,dt - (1+t)(1+t)3\,dt + (3t)^2\,dt = -\frac{5}{3}$$

(c) $$\int_C x^2yz\,ds \;\underset{\begin{cases} x=\cos t \\ y=\cos t \\ z=\sqrt{2}\sin t \end{cases}\; 0\le t\le \frac{\pi}{2}}{=\!=}\; \int_0^{\frac{\pi}{2}} \cos^2 t \cos t \sqrt{2}\sin t \sqrt{\sin^2 t + \sin^2 t + 2\cos^2 t}\,dt$$

$$= 2\int_0^{\pi/2} \cos^3 t \sin t\,dt = -\frac{2\cos^4}{4}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

(d) $$\int_C u_t\,ds = \int_C 2xy^2z\,dx + 2x^2yz\,dy + x^2y^2\,dz$$

$(x = \cos t \quad y = \sin t \quad z = 2 \quad 0\le t\le 2\pi)$

$$= \int_0^{2\pi} (2\cos t\sin^2 t(-\sin t) + 2\cos^2 t \sin t \cos t + \cos^2 t \sin^2 t \cdot 0)\,dt$$

$$= \int_0^{2\pi} (-2\cos t\sin^3 t + 2\cos^3 t \sin t)\,dt$$

$$= \int_0^{2\pi} 2\cos t \sin t \cos 2t\,dt$$

$= 0$

(e) $\int_C u_t \, ds$, $\vec{u} = \text{curl}\,[y^2 i + z^2 j + x^2 k]$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$= [-2z, \ -2x, \ -2y] \qquad \begin{cases} x = 2t+1 \\ y = t^2 \\ z = t^3+1 \end{cases} \quad 0 \le t \le 1$$

$$\int_C u_t \, ds = -2\int_C z\,dx + x\,dy + y\,dz = -2\int_0^1 \{[t^3+1]2 + (2t+1)2t + t^2 \cdot 3t^2\}\,dt$$

$$= -2\int_0^1 3t^4 + 2t^3 + 4t^2 + 2t + 2 \ dt = -\frac{163}{15}$$

2 (a) $\vec{u} = \text{grad}\,F$

Show $\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} u_t \, ds = F(x_2, y_2, z_2) - F(x_1, y_1, z_1)$

$\vec{u} = Pi + Qj + Rk$, $\nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$

If $\vec{u} = \nabla F$ then $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$, $R = \frac{\partial F}{\partial z}$

So $\int_{A=(x_1, y_1, z_1)}^{B=(x_2, y_2, z_2)} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = \int dF = F(x_2, y_2, z_2) - F(x_1, y_1, z_1)$

(b) $\int_C u_t \, ds = \int_C \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = \int dF = F(x, y, z) - F(x, y, z)$

$= 0$

(because the path is closed)