1. (a) $2x^2+y^2-z^2=3$ (2.71) $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

H/wk no. 3

$F(x,y,z) = 2x^2+y^2-z^2-3$

$F_x = 4x$ $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z} = -\frac{4x}{-2z} = \frac{2x}{z}$

$F_y = 2y$

$F_z = -2z$, $\left(\frac{\partial z}{\partial y}\right)_x = \frac{y}{z}$

(b) $F(x,y,z) = xyz + 2x^2z + 3xz^2 - 1$

$F_x = yz + 4xz + 3z^2$ $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{yz+4xz+3z^2}{xy+2x^2+6xz}$

$F_y = xz$

$F_z = xy + 2x^2 + 6xz$ $\left(\frac{\partial z}{\partial y}\right)_x = \frac{-z}{y+2x+6z}$

(c) $F(x,y,z) = z^3 + xz + 2yz - 1$

$F_x = z$ $\left(\frac{\partial z}{\partial x}\right)_y = \frac{-z}{3z^2+x+2y}$

$F_y = 2z$ $\left(\frac{\partial z}{\partial y}\right)_x = \frac{-2z}{3z^2+x+2y}$

$F_z = 3z^2+x+2y$

(d) $F(x,y,z) = e^{xz} + e^{yz} + z - 1$

$F_x = ze^{xz}$ $\left(\frac{\partial z}{\partial x}\right)_y = \frac{-ze^{xz}}{xe^{xz}+ye^{yz}+1}$

$F_y = ze^{z}$

$F_z = xe^{xz} + ye^{yz} + 1$ $\left(\frac{\partial z}{\partial y}\right)_x = \frac{-ze^{yz}}{xe^{xz}+ye^{yz}+1}$

3. (a) $F = x^2 - y^2 + u^2 + 2v^2 - 1$

$G = x^2 + y^2 - u^2 - v^2 - 2$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{\partial(F\ G)}{\partial(y\ v)} \Big/ \frac{\partial(F\ G)}{\partial(u,v)} = \begin{vmatrix} -2y & 4v \\ 2y & -2v \end{vmatrix} \Big/ \begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix} = \frac{y}{u}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\partial(F\ G)}{\partial(x\ v)} \Big/ \frac{\partial(F\ G)}{\partial(u\ v)} = \begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix} \Big/ \begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix} = \frac{3x}{u}$$

(b) $F_1 = e^u + xu - yv - 1 \qquad G = e^v - xv + yu - 2$

$$\left(\frac{\partial u}{\partial x}\right)_x = \begin{vmatrix} -v & -y \\ u & e^v - x \end{vmatrix} \Big/ \begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix} = \frac{ve^v - xv - uy}{e^{u+v} - xe^u + xe^v - x^2 + y^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \begin{vmatrix} u & -y \\ -v & e^v - x \end{vmatrix} \Big/ \begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix} = \frac{xu + vy - ue^v}{e^{u+v} - xe^u + xe^v - x^2 + y^2}$$

(c) $F_1 = x^2 + xu - yv^2 + uv - 1 \qquad G = xu - 2yv - 1$

$$\left(\frac{\partial u}{\partial x}\right)_x = \begin{vmatrix} -v^2 & -2yv + u \\ -2v & -2y \end{vmatrix} \Big/ \begin{vmatrix} x + v & -2yv + u \\ x & -2y \end{vmatrix} = \frac{2uv - 2yv^2}{2xy + 2vy + xu - 2xyv}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \begin{vmatrix} 2x + u & -2yv + u \\ u & -2y \end{vmatrix} \Big/ \begin{vmatrix} x + v & -2yv + u \\ x & -2y \end{vmatrix} = \frac{-(4xy + 2yu + u^2 - 2yuv)}{2xy + 2vy + xu - 2xyv}$$

6. $$\left(\frac{\partial F_i}{\partial x_i}\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 5 & 1 & -1 & 4 \end{bmatrix}$$

(a) $$\left(\frac{\partial x_1}{\partial x_3}\right)_{x_4} = -\frac{\partial(F_1\ F_2)}{\partial(x_3\ x_2)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_1\ x_2)} = \frac{1}{2}$$

$$\left(\frac{\partial x_1}{\partial x_4}\right)_{x_3} = -\frac{\partial(F_1\ F_2)}{\partial(x_4\ x_2)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_1\ x_2)} = -1$$

(b) $$\left(\frac{\partial x_1}{\partial x_3}\right)_{x_2} = -\frac{\partial(F_1\ F_2)}{\partial(x_3\ x_4)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_1\ x_4)} = -1$$

$$\left(\frac{\partial x_4}{\partial x_3}\right)_{x_2} = -\frac{\partial(F_1\ F_2)}{\partial(x_1\ x_3)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_1\ x_4)} = \frac{3}{2}$$

(c) $$\frac{\partial(x_1\ x_2)}{\partial(x_3\ x_4)} = -\frac{\partial(F_1\ F_2)}{\partial(x_3\ x_4)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_1\ x_2)} = -1$$

$$\frac{\partial(x_3\ x_4)}{\partial(x_1\ x_2)} = -\frac{\partial(F_1\ F_2)}{\partial(x_1\ x_2)} \Big/ \frac{\partial(F_1\ F_2)}{\partial(x_3\ x_4)} = -1.$$

1. $x = r\cos\theta \qquad y = r\sin\theta$

(a) $dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$\because \quad dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta \;\lrcorner$

$dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$\because \quad dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\, d\theta + r\cos\theta\, d\theta \;\lrcorner$

(b) $dr = \cos\theta\, dx + \sin\theta\, dy \quad \because \; dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy, \qquad F = x - r\cos\theta, \; G = y - r\sin\theta$

$\frac{\partial r}{\partial x} = -\frac{\partial(F\,G)}{\partial(x\,\theta)} \Big/ \frac{\partial(F\,G)}{\partial(r\,\theta)} = \cos\theta = \frac{\partial r}{\partial x}$

$\frac{\partial r}{\partial y} = \frac{-\partial(F\,G)}{\partial(y\,\theta)} \Big/ \frac{\partial(F\,G)}{\partial(r\,\theta)} = \sin\theta = \frac{\partial r}{\partial y}$

$\Rightarrow dr = \cos\theta\, dx + \sin\theta\, dy \;\lrcorner$

$d\theta = -\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy \quad \because \; d\theta = \frac{\partial\theta}{\partial x} dx + \frac{\partial\theta}{\partial y} dy$

$\frac{\partial\theta}{\partial x} = \frac{-\partial(F\,G)}{\partial(r\,x)} \Big/ \frac{\partial(F\,G)}{\partial(r\,\theta)} = \frac{-\sin\theta}{r}$

$\frac{\partial\theta}{\partial y} = \frac{\partial(F\,G)}{-\partial(r\,y)} \Big/ \left.\frac{\partial(F\,G)}{\partial(r\,\theta)}\right| = \frac{\cos\theta}{r}$

$\Rightarrow d\theta = -\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy \;\lrcorner.$

(c)

$\because$

$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{-\partial(F,G)}{\partial(r\,y)} \Big/ \frac{\partial(F\,G)}{\partial(x\,y)} = \cos\theta$

$\left(\frac{\partial x}{\partial r}\right)_y = -\frac{\partial(F\,G)}{\partial(r\,\theta)} \Big/ \frac{\partial(F\,G)}{\partial(x,\theta)} = \sec\theta$

$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{\partial(F\,G)}{\partial(x\,\theta)} \Big/ \frac{\partial(F\,G)}{\partial(r\,\theta)} = \cos\theta$

$\left(\frac{\partial r}{\partial x}\right)_\theta = -\frac{\partial(F\,G)}{\partial(y\,x)} \Big/ \frac{\partial(F\,G)}{\partial(y\,r)} = \sec\theta \;\lrcorner$

(d) $\frac{\partial(x,y)}{\partial(r,\theta)} = r \quad \because \; \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \left(\frac{\partial x}{\partial r}\right)_\theta & \left(\frac{\partial x}{\partial \theta}\right)_r \\ \left(\frac{\partial y}{\partial r}\right)_\theta & \left(\frac{\partial y}{\partial \theta}\right)_r \end{vmatrix} = r \;\lrcorner$

$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} \quad \because \; \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r} \;\lrcorner$

3. $X = u^2 - v^2$ $\quad Y = 2uv$ $\quad$ let $F = u^2 - v^2 - x$ $\quad G = 2uv - y$

(a) $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 4u^2 + 4v^2$

(b)

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\partial(F\ G)}{\partial(x\ v)} \Big/ \frac{\partial(F\ G)}{\partial(u\ v)} = \frac{u}{2u^2 + 2v^2}$$

$$\left(\frac{\partial v}{\partial x}\right)_y = -\frac{\partial(F, G)}{\partial(u, x)} \Big/ \frac{\partial(F, G)}{\partial(u, v)} = \frac{-v}{2u^2 + 2v^2}$$

5. $x = f(u, v, w)$ $\quad y = g(u, v, w)$ $\quad z = h(u, v, w)$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix} = \frac{\partial(f\ g\ h)}{\partial(u, v, w)} \overset{\text{let}}{=} \frac{\partial(F\ G\ H)}{\partial(u, v, w)}$$

where $F = f(u, v, w)^{-1}$
$G = g(u\ v\ w)^{-1}$
$H = h(u, v\ w)^{-1}$

$$\frac{\partial u}{\partial x} = -\frac{\partial(F\ G\ H)}{\partial(x\ v\ w)} \Big/ \frac{\partial(F\ G\ H)}{\partial(u\ v\ w)} = -\begin{vmatrix} F_x & F_v & F_w \\ G_x & G_v & G_w \\ H_x & H_v & H_w \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$= \frac{1}{J} \begin{vmatrix} g_v & g_w \\ h_v & h_w \end{vmatrix} = \frac{1}{J} \frac{\partial(y, z)}{\partial(v, w)}$$

Other things are same.