

Math 285 Section C1 Exam 2 (SOLUTIONS)

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Problem 1. [16 points] Select the correct answer for each of the following questions. Each question has exactly one correct answer. **You do not need to provide any explanations in this problem.**

(1) Let $f(t)$ be a 6-periodic function defined as

$$f(t) = \begin{cases} -20t^9 \cos(5t), & \text{if } -3 < t < 0, 0 < t < 3, \\ 1, & \text{if } t = 0, 3, -3. \end{cases}$$

Let a_n, b_n be the general Fourier series coefficients for $f(t)$. Then:

- (a) We have $b_n = 0$ for all $n \geq 1$.
- (b) We have $3a_2 - 25a_7 = 0$.
- (c) We have $a_n = \frac{2}{3} \int_0^3 f(t) \cos(\frac{\pi nt}{3}) dt$ for all $n \geq 0$.
- (d) None of the above.

Answer: B

(2) Let $f(t) = 5t$ for $0 < t < 3$. Let a_n be the Fourier Cosine Series coefficients for $f(t)$. Then:

- (a) We have $7.5 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$.
- (b) We have $20 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(4\pi n/3)$.
- (c) We have $a_n = 0$ for all $n \geq 0$.
- (d) None of the above.

Answer: D

(3) Let $f(t)$ and a_n be as in part (2). Then

- (a) We have $5 = \sum_{n=1}^{\infty} -a_n(\pi n/3) \sin(\pi nt/3)$ for each $t \in (-\infty, \infty)$.
- (b) We have $5 = \sum_{n=1}^{\infty} -a_n(\pi n/3) \sin(\pi nt/3)$ for each $t \in (0, 3)$.
- (c) We have $5 \neq \sum_{n=1}^{\infty} -a_n(\pi n/3) \sin(\pi nt/3)$ for each $t \in (-\infty, \infty)$.
- (d) None of the above.

Answer: B

(4) Consider an eigenvalue problem

$$\begin{cases} y'' + p(x)y' + \lambda q(x)y = 0 \\ y(0) = y(1) = 0, \end{cases}$$

where $p(x), q(x)$ are continuous on $[0, 1]$. Let λ_* be an eigenvalue of this problem. Then:

- (a) With $\lambda = \lambda_*$ the above system has only the zero solution on $[0, 1]$.
- (b) With $\lambda = \lambda_*$ the above system has at least two different solutions on $[0, 1]$.
- (c) With $\lambda = \lambda_*$ the above system has exactly two different solutions on $[0, 1]$.
- (d) None of the above.

Answer: B

Problem 2.[20 points]

Use the Method of Undetermined Coefficients to find the general solution of the equation $y'' - 3y' = 3x$ on the interval $(-\infty, \infty)$. Give all the details of your work.

Solution.

The complimentary homogeneous equation is $y'' - 3y' = 0$. Its characteristic equation is $r^2 - 3r = 0$, that is $r(r - 3) = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = 3$. Therefore the general solution of $y'' - 3y' = 0$ is:

$$y_c = c_1 e^{0x} + c_2 e^{3x} = c_1 + c_2 e^{3x},$$

where c_1, c_2 are arbitrary constants.

The Method of Undetermined Coefficients dictates that we look for a particular solution y_p of the equation $y'' - 3y' = 3x$ in the form:

$$y = x(A + Bx),$$

where the x factor is needed to eliminate duplication.

Thus

$$y = Ax + Bx^2, \quad y' = A + 2Bx, \quad y'' = 2B.$$

Substituting this in the equation $y'' - 3y' = 3x$ we get:

$$2B - 3(A + 2Bx) = 3x$$

$$(2B - 3A) - 6Bx = 3x$$

$$2B - 3A = 0, \quad -6B = 3$$

$$B = -\frac{1}{2}, \quad A = \frac{2}{3}B = -\frac{1}{3}.$$

Thus $y_p = -\frac{1}{3}x - \frac{1}{2}x^2$ is a particular solution of $y'' - 3y' = 3x$. The general solution of this equation is:

$$y = y_c + y_p = c_1 + c_2 e^{3x} - \frac{1}{3}x - \frac{1}{2}x^2,$$

where c_1, c_2 are arbitrary constants.

Problem 3.[22 points] Find all the nonnegative eigenvalues and the corresponding eigenfunctions for the eigenvalue problem:

$$(\dagger) \quad \begin{cases} y'' + \lambda y = 0 \\ y(0) = y'(2) = 0, \end{cases}$$

Give all the details of your work.

Solution.

1) We first check if $\lambda = 0$ is an eigenvalue. With $\lambda = 0$ the main equation takes the form $y'' = 0$. Its characteristic equation is $r^2 = 0$ with a repeated real root $r_{1,2} = 0$ and the general solution $y = A + Bx$. Then $y' = B$. From $y(0) = 0$ we get $A + B \cdot 0 = A = 0$. Thus $y = Bx$. From $y'(2) = 0$ we get $B = 0$. Thus $y = 0$ is the only solution of the system (\dagger) with $\lambda = 0$ and hence $\lambda = 0$ is not an eigenvalue.

2) Suppose now $\lambda > 0$. We can then uniquely represent λ as $\lambda = \alpha^2$, where $\alpha > 0$. The equation $y'' + \lambda y = 0$ becomes $y'' + \alpha^2 y = 0$. Its characteristic equation is $r^2 + \alpha^2 = 0$, which has two complex conjugate roots $r_{1,2} = \pm \alpha i$. Therefore the general solution of $y'' + \alpha^2 y = 0$ is:

$$y = A \cos \alpha x + B \sin \alpha x.$$

By differentiating we get:

$$y' = -A\alpha \sin \alpha x + B\alpha \cos \alpha x.$$

From $y(0) = 0$ we get $A \cos 0 + B \sin 0 = A \cdot 1 + B \cdot 0 = A = 0$. Thus $y = B \sin \alpha x$ and $y' = B\alpha \cos \alpha x$.

From $y'(2) = 0$ we get

$$B\alpha \cos 2\alpha = 0$$

If $\cos 2\alpha \neq 0$ then $B = 0$ and so $y = 0$, in which case $\lambda = \alpha^2$ is not an eigenvalue of (\dagger) .

If $\cos 2\alpha = 0$ then with an arbitrary B the function $y = B \sin \alpha x$ is a solution of (\dagger) , so $\lambda = \alpha^2$ is an eigenvalue.

The conditions $\alpha > 0$ and $\cos 2\alpha = 0$ yield $2\alpha = \frac{\pi}{2} + \pi n$, where $n \geq 0$. Thus $\alpha_n = \frac{\pi}{4} + \frac{\pi n}{2} = \frac{(2n+1)\pi}{4}$, where $n = 0, 1, 2, \dots$

This gives us eigenvalues

$$\lambda_n = \alpha_n^2 = \frac{(2n+1)^2 \pi^2}{16}, \quad n = 0, 1, 2, 3, \dots$$

The corresponding eigenfunctions are $y_n = \sin \alpha_n x$:

$$y_n = \sin \frac{(2n+1)\pi x}{4}, \quad n = 0, 1, 2, 3, \dots$$

Problem 4.[20 points]

Find the Fourier Cosine Series of the function $f(t) = 2t$, $0 < t < 3$. Give all the details of your work.

Solution.

We have $L = 3$ and therefore

$$a_0 = \frac{2}{L} \int_0^L f(t) dt = \frac{2}{3} \int_0^3 2t dt = \frac{2}{3} [t^2]_0^3 = \frac{2}{3} \cdot 9 = 6.$$

For $n > 0$ we have

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(t) \cos \frac{\pi n t}{L} dt = \frac{2}{3} \int_0^3 2t \cos \frac{\pi n t}{3} dt = \\ &= \frac{9}{\pi^2 n^2} \frac{4}{3} \int_0^3 \frac{\pi n t}{3} \cos \frac{\pi n t}{3} d \frac{\pi n t}{3} = \quad \left[\text{with } u = \frac{\pi n t}{3} \right] \\ &= \frac{12}{\pi^2 n^2} \int_0^{\pi n} u \cos u du = \frac{12}{\pi^2 n^2} [u \sin u + \cos u]_0^{\pi n} = \\ &= \frac{12}{\pi^2 n^2} [\cos(\pi n) - 1] = \frac{12}{\pi^2 n^2} [(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{24}{\pi^2 n^2}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Therefore the Fourier Cosine Series is:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n t}{L} = 3 - \frac{24}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{\pi n t}{3}.$$

Problem 5. [22 points]

Let $f(x)$ be a function defined for $0 < x < 5$ such that the Fourier Sine series of $f(x)$ is

$$\frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi n x}{5}$$

and the Fourier Cosine Series of $f(x)$ is

$$5 - \frac{40}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{\pi n x}{5}$$

Find a trigonometric series solution of the following boundary value problem (note that some of the information provided above may be redundant):

$$y'' - 2y = f(x), \quad y'(0) = 0, y'(5) = 0.$$

Solution.

Since the functions $y = \sin \frac{\pi n x}{5}$ do not satisfy the endpoint conditions $y'(0) = y'(5) = 0$, while the functions $y = \cos \frac{\pi n x}{5}$ do satisfy this condition, we will look for the solution of our system in the form

$$y = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{\pi n x}{5}.$$

Via term-wise differentiation we get

$$y' = \sum_{n=1}^{\infty} -c_n \frac{\pi n}{5} \sin \frac{\pi n x}{5}$$

$$y'' = \sum_{n=1}^{\infty} -c_n \frac{\pi^2 n^2}{5^2} \cos \frac{\pi n x}{5}$$

Substituting this into the equation $y'' - 2y = f(x)$ we get

$$-2c_0 + \sum_{n=1}^{\infty} \left[-c_n \frac{\pi^2 n^2}{5^2} - 2c_n \right] \cos \frac{\pi n x}{5} = 5 - \frac{40}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{\pi n x}{5}$$

$$c_0 = -\frac{5}{2}$$

$$c_n = \begin{cases} 0, & n \text{ even, positive} \\ \frac{-\frac{40}{\pi^2 n^2}}{\frac{\pi^2 n^2}{5^2} + 2} = \frac{1000}{n^2 \pi^2 (n^2 \pi^2 + 50)}, & n \text{ odd} \end{cases}$$

Therefore

$$y = -\frac{5}{2} + \sum_{n \text{ odd}} \frac{1000}{n^2 \pi^2 (n^2 \pi^2 + 50)} \cos \frac{\pi n x}{5}$$