Math 295E First Midterm Exam (SOLUTIONS); February 20, 2004

Problem 1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $A \subseteq \mathbb{R}^n$ be a compact subset. Prove that there exists $x_0 \in A$ such that

$$f(x_0) = \sup_{x \in A} f(x).$$

Solution.

Since f is continuous and A is compact, it follows that $f(A)$ is a compact subset of $\mathbb{R}$. Thus it suffices to show that any compact subset B of $\mathbb{R}$ contains $\sup B$.

Let $B \subseteq \mathbb{R}$ be compact. Since B is compact, it is closed and bounded. Hence $b = \sup B \in \mathbb{R}$ exists. Suppose that $b \notin B$. Since $\mathbb{R} - B$ is open, there is some $\epsilon > 0$ such that $(b - \epsilon, b + \epsilon) \subseteq \mathbb{R} - B$. Since $b \geq x$ for every $x \in B$, this implies that $b - \frac{\epsilon}{2} \geq x$ for every $x \in B$. Thus $b - \frac{\epsilon}{2}$ is an upper bound for B that is smaller than b. This contradicts our assumption that $b = \sup B$.

Problem 2.

Let $f : [0, 1] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x \neq 0 \text{ is rational, } x = \frac{p}{2^n} \text{in lowest terms} \\ 0, & \text{if } x \neq 0 \text{ is rational , } x = \frac{p}{q} \text{ in lowest terms, where } q \text{ in not a power of 2} \\ 0, & \text{if } x = 0, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is integrable on $[0, 1]$.

Solution.

We will show that for every irrational point $x \in [0, 1]$ the function f is continuous at x. By definition, $f(x) = 0$. Suppose $x_k \in [0, 1]$ is a sequence such that $\lim_{k\to\infty} x_k = x$.

We claim that $\lim_{n\to\infty} f(x_n) = 0$. This would imply the continuity of f at x.

Clearly, it is enough to prove this claim assuming that each x_k in the sequence is a rational number of the form $x_k = \frac{p_k}{2^{n_k}}$, in lowest terms. Since x is irrational and $0 \leq p_k \leq 2^{n_k}$, we must have $\lim_{k\to\infty} 2^{n_k} = \infty$ [for otherwise the sequence x_k would contain a constant subsequence of rational numbers and hence could not have an irrational limit]. Hence $\lim_{n\to\infty} n_k = \infty$ and therefore $\lim_{n\to\infty} f(x_k) = \frac{1}{n_k} = 0$, as claimed.

Note that by definition $|f(x)| \leq 1$ for every $x \in [0, 1]$. Thus $f : [0, 1] \to \mathbb{R}$ is a bounded function whose discontinuity set is contained in $[0, 1] \cap \mathbb{Q}$. Therefore f is integrable on $[0, 1]$ by Theorem 3-8.

Problem 3.

Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for every $v \in \mathbb{R}^2$

$$f(0, 0) = \lim_{t\to 0} f(tv)$$

but $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Justify why your function has the required properties.

Solution

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x,y) = \begin{cases} 1, & \text{if } y = x^2 \text{ and } x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then for any nonzero vector $v = (a,b) \in \mathbb{R}$ there is $\epsilon > 0$ such that $|bt| \neq |at|^2$ for every $t \in \mathbb{R}$ with $0 < |t| < \epsilon$. Hence $\lim_{t\to 0} f(tv) = 0 = f(0,0)$. For $v = (0,0)$ we have $tv = 0$ for all $t \in \mathbb{R}$ and hence again $\lim_{t\to 0} f(tv) = 0 = f(0,0)$.

On the other hand, any open rectangle containing $(0,0)$ contains a point (x, x^2) with $x \neq 0$, for which $f(x,x^2) = 1 \neq 0$. Therefore the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Problem 4.

Let $f : [0,2\pi] \times [0,2\pi] \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ \sin y & \text{otherwise.} \end{cases}$$

Is it true that the equality

$$\int_{[0,2\pi]\times[0,2\pi]} f = \int_0^{2\pi} (\int_0^{2\pi} f(x,y)\, dy) dx$$

holds?

If yes, explain why. If not, explain why not and why Fubini's Theorem is not applicable here.

Solution.

No, the above equality does not hold.

By definition of f if x is rational then $\int_0^{2\pi} f(x,y)\,dy = \int_0^{2\pi} 0\,dy = 0$ and if x is irrational then $\int_0^{2\pi} f(x,y)\,dy = \int_0^{2\pi} \sin y\,dy = 0$. Hence $\int_0^{2\pi}(\int_0^{2\pi} f(x,y)\,dy)dx = \int_0^{2\pi} 0\,dx = 0$.

On the other hand we will prove that f is not integrable (which explains why Fubini's theorem is not applicable here). Indeed let

$$S = \{(x,y) \in [0,2\pi] \times [0,2\pi] : y \notin \{0, \pi, 2\pi\}\}.$$

We claim that f is discontinuous at every point of S. Indeed, if $(x,y) \in S$ then $\sin y \neq 0$. Since both rational and irrational points are dense in $[0,2\pi]$, it follows that there exist sequences $x_n \in [0,2\pi] \cap \mathbb{Q}$ and $x_n' \in [0,2\pi] - \mathbb{Q}$ that converge to x. Thus both (x_n, y) and (x_n', y) converge to (x,y). We have $\lim_{n\to\infty} f(x_n,y) = \lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} f(x_n',y) = \lim_{n\to\infty} \sin y = \sin y \neq 0$. Hence f is discontinuous at (x,y) as claimed.

The set S does not have measure zero since it contains an open rectangle (for example $(0,1)\times(0,1) \subseteq S$). Therefore f is not integrable on $[0,2\pi]\times[0,2\pi]$.