Math 295E Second Midterm Exam; April 16, 2004 (SOLUTIONS)

**Problem 1.**

Consider the 1-forms $\omega_1 = xy\,dx - z^2dz$ and $\omega_2 = (2x + z + y)\,dy$ on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $f(t, s) = (t + s, e^{t-s}, ts)$.

(1) Compute the forms $\omega_1 \wedge \omega_2$ and $d(\omega_1 \wedge \omega_2)$.
(2) Compute the form $f^*(d(\omega_1 \wedge \omega_2))$.
(3) Consider the singular 1-cube $c : [0, 1] \to \mathbb{R}^3$ defined as $c(t) = (1, t, t^2)$. Compute $\int_c \omega_1$.

**Solution.**

1) We have

$$\omega_1 \wedge \omega_2 = (xy\,dx - z^2dz) \wedge (2x + z + y)\,dy =$$
$$(2x^2y + xyz + xy^2)dx \wedge dy - (2xz^2 + z^3 + z^2y)dz \wedge dy =$$
$$(2x^2y + xyz + xy^2)dx \wedge dy + (2xz^2 + z^3 + z^2y)dy \wedge dz.$$

Hence

$$d(\omega_1 \wedge \omega_2) =$$
$$\frac{\partial}{\partial z}(2x^2y + xyz + xy^2)dz \wedge dx \wedge dy + \frac{\partial}{\partial x}(2xz^2 + z^3 + z^2y)dx \wedge dy \wedge dz =$$
$$xy\,dz \wedge dx \wedge dy + 2z^2\,dx \wedge dy \wedge dz = xy\,dx \wedge dy \wedge dz + 2z^2\,dx \wedge dy \wedge dz =$$
$$(xy + 2z^2)dx \wedge dy \wedge dz$$

2) Since $f^*(d(\omega_1 \wedge \omega_2))$ is a differential 3-form on $\mathbb{R}^2$, we have

$$f^*(d(\omega_1 \wedge \omega_2)) = 0.$$

3) We have:

$$\int_c \omega_1 = \int_{[0,1]} c^*\omega_1 = \int_{[0,1]} -(t^2)^2\,dt^2 =$$
$$= -\int_{[0,1]} 2t^5\,dt = -\int_0^1 2t^5\,dt = -[t^6/3]_0^1 = -1/3.$$

**Problem 2.**

Let $M = \{(x, y, x^2 - y) \in \mathbb{R}^2 | x, y \in \mathbb{R}\}$. Then $M$ is a 2-manifold in $\mathbb{R}^3$ by Problem 5-6 and the map $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x, y, x^2 - y)$, is a coordinate system on $M$.

Let $p = (0, 0, 0) \in M$ and let $v = (1, 1, 5) \in \mathbb{R}^3$ so that $v_p \in \mathbb{R}^3_p$.

Does $v_p$ belong to $M_p$? Provide a proof justifying your answer.

**Solution.**

We have

$$f'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2x & -1 \end{bmatrix}$$

Thus

$$A := f'(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

By definition $M_p$ consists of all vectors $(Aw)_p$ where $w \in \mathbb{R}^2$.
For any $(w_1, w_2) \in \mathbb{R}^2$ we have:

$$A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ -w_2 \end{bmatrix}$$

.

Suppose $v_p = (1,1,5)_p \in M_p$. Then there exists $w = (w_1, w_2) \in \mathbb{R}^2$ such that $(w_1, w_2, -w_2) = (1,1,5)$. Obviously, this is impossible, since $1 \neq -5$.

Therefore $v_p \notin M_p$.

**Problem 3.**
For $n \geq 1$ let $SL(n,\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$ with determinant 1. Let $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of all $n \times n$ matrices over $\mathbb{R}$. Prove that $SL(n,\mathbb{R}) \subseteq M(n,\mathbb{R})$ is a manifold and find the dimension of this manifold.

**Solution.** We identify $M(n,\mathbb{R})$ with $\mathbb{R}^{n^2}$ and denote the coordinate functions on $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ by $x_{ij}$, where $1 \leq i,j \leq n$. Consider the function $f : M(n,\mathbb{R}) \to \mathbb{R}$ defined as $f(X) := det(X) - 1$, where $X \in M(n,\mathbb{R})$. Then $SL(n,\mathbb{R}) = f^{-1}(0)$.

The function $f(X) := det(X) - 1$ is polynomial in the coordinates $x_{ij}$ of $X$ and therefore is smooth. We claim that for any $X \in SL(n,\mathbb{R})$ the derivative $f'|_X : M(n,\mathbb{R}) \to \mathbb{R}$ is surjective, that is $f'|_X \neq 0$. By Theorem 5.1 this would imply that $SL(n,\mathbb{R}) \subseteq M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ is a manifold of dimension $n^2 - 1$.

Note that if $g_{ij}(t)$, where $1 \leq i,j \leq n$, are differentiable functions of one variable $t$ and

$$G(t) = \begin{vmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{vmatrix}$$

then

$$G'(t) = \begin{vmatrix} g'_{11} & \dots & g'_{1n} \\ g_{21} & \dots & g_{2n} \\ \vdots & \vdots & \vdots \\ g_{n1} & \dots & g_{nn} \end{vmatrix} + \begin{vmatrix} g_{11} & \dots & g_{1n} \\ g'_{21} & \dots & g'_{2n} \\ \vdots & \vdots & \vdots \\ g_{n1} & \dots & g_{nn} \end{vmatrix} + \dots + \begin{vmatrix} g_{11} & \dots & g_{1n} \\ g_{21} & \dots & g_{2n} \\ \vdots & \vdots & \vdots \\ g'_{n1} & \dots & g'_{nn} \end{vmatrix}$$

Therefore for

$$\frac{\partial f}{\partial x_{ij}}|_X = \frac{\partial}{\partial x_{ij}} \begin{vmatrix} x_{11} & \dots & x_{1j} & \dots & x_{1n} \\ x_{21} & \dots & x_{2j} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i1} & \dots & x_{ij} & \dots & x_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nj} & \dots & x_{nn} \end{vmatrix} = \begin{vmatrix} x_{11} & \dots & x_{1j} & \dots & x_{1n} \\ x_{21} & \dots & x_{2j} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nj} & \dots & x_{nn} \end{vmatrix} = (adjX)_{ji}$$

where $adjX$ is the adjoint matrix of $X$.

If $X \in SL(2, \mathbb{R})$ then $det(X) = 1$ and hence $X$ is invertible. Moreover, in this case $X^{-1} = \frac{1}{det(X)} adjX = adjX \neq 0$. Therefore there exist $i, j \in \{1, \dots, n\}$ such that

$$0 \neq (X^{-1})_{ij} = (adjX)_{ji} = \frac{\partial f}{\partial x_{ji}}|_X.$$

Thus $f'|_X \neq 0$, as required.

**Problem 4.**

Let $V$ be a vector space over $\mathbb{R}$. A $k$-tensor $T : V^k \to \mathbb{R}$ is called *symmetric* if $T(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$ for every permutation $\sigma \in S_k$ and for all $v_1, \dots, v_k \in V$. Let $\mathcal{S}_k(V)$ be the set of all symmetric $k$-tensors on $V$ (so that $\mathcal{S}_k(V)$ is a subspace of the space $\mathcal{T}_k(V)$ of all $k$-tensors on $V$).

Construct linear maps $Sym_k : \mathcal{T}_k(V) \to \mathcal{S}_k(V)$, $k \geq 1$, such that $Sym_k(T) = T$ for any symmetric $k$-tensor $T$ (with $k \geq 1$) and $Sym_k(T) = 0$ for any alternating $k$-tensor $T$ (with $k \geq 2$). Prove that the maps you constructed have the required properties.

**Solution.**

For a $k$-tensor $T$ and $v_1, \dots, v_k \in V$ put

$$Sym_k(T)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

It is clear that $Sym_k$ is a linear map from $\mathcal{T}_k(V)$ to $\mathcal{T}_k(V)$.

1) We claim that if $k \geq 1$ then for any $T \in \mathcal{T}_k(V)$ the tensor $Sym_k(T)$ is symmetric, that is $Sym_k(T) \in \mathcal{S}_k(V)$.

Indeed, let $T \in \mathcal{T}_k(V)$ be arbitrary. Let $\eta \in S_k$ be an arbitrary permutation.

Then for any $v_1, \dots, v_k \in V$ we have

$$Sym_k(T)(v_{\eta(1)}, \dots, v_{\eta(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma\eta(1)}, \dots, v_{\sigma\eta(k)}) =$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k, \sigma' = \sigma\eta} T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)}) =$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)}) = Sym_k(T)(v_1, \dots, v_k).$$

Thus $Sym_k(T)$ is symmetric, as required.

2) Suppose that $k \geq 1$ and $T$ is a symmetric $k$-tensor.

Then for any any $v_1, \dots, v_k \in V$ we have:

$$Sym_k(T)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} T(v_1, \dots, v_k) =$$

$$\frac{1}{k!} k! T(v_1, \dots, v_k) = T(v_1, \dots, v_k).$$

Thus $Sym_k(T) = T$, as required.

3) Suppose $k \geq 2$ and $T$ is an alternating $k$-tensor.

Let $\tau := (12) \in S_k$ be the transposition interchanging 1 and 2.

Then for any any $v_1, \dots, v_k \in V$ we have:

$$Sym_k(T)(v_1, v_2, v_3, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \dots, v_{\sigma(k)}) =$$

$$-\frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma(2)}, v_{\sigma(1)}, v_{\sigma(3)}, \dots, v_{\sigma(k)} =$$

$$-\frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma\tau(1)}, v_{\sigma\tau(1)}, v_{\sigma\tau(3)}, \dots, v_{\sigma\tau(k)} =$$

$$-\frac{1}{k!} \sum_{\sigma \in S_k, \sigma' = \sigma\tau} T(v_{\sigma'(1)}, v_{\sigma'(2)}, v_{\sigma'(3)}, \dots, v_{\sigma'(k)}) =$$

$$= -\frac{1}{k!} \sum_{\sigma' \in S_k} T(v_{\sigma'(1)}, v_{\sigma'(2)}, v_{\sigma'(3)}, \dots, v_{\sigma'(k)}) = -Sym_k(T)(v_1, v_2, v_3, \dots, v_k).$$

Thus $Sym_k(T)(v_1, v_2, v_3, \dots, v_k) = -Sym_k(T)(v_1, v_2, v_3, \dots, v_k)$ and therefore $Sym_k(T)(v_1, v_2, v_3, \dots, v_k) = 0$. Hence $Sym_k(T) = 0$, as required.