Math 285 Section A1 Exam 1 (SOLUTIONS)
Prof. I.Kapovich October 6, 2003

Problem 1. [16 points] Select the correct answer for each of the following questions. Each question has exactly one correct answer. You do not need to provide any explanations in this problem.

(1) For the initial value problem $\frac{dy}{dx} = cos(x^2 + y^4)$, $y(1) = 4$ on the interval $I = (-\infty, \infty)$

(a) A solution is guaranteed to exist on I.
(b) A solution is guaranteed to exist on an interval $(1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$.
(c) A unique solution is guaranteed to exist on the interval I.
(d) There are infinitely many solutions on I.

Answer: (b)

(2) The differential equation $y'' - 2xy' + x^3 y = 0$ on the interval $I = (-\infty, \infty)$

(a) has exactly one solution;
(b) has the property that for any solutions y_1, y_2 the function $y_1 + y_2$ is also a solution.
(c) has characteristic equation $r^2 - 2r + 1 = 0$.

Answer (b)

(3) Making the substitution $v = y/x$ in an equation $y' = F(y/x)$ transforms this differential equation into

(a) a linear first order differential equation;
(b) a separable differential equation;
(c) an exact equation;
(d) a homogeneous equation;

Answer: (b)

(4) The equation $(x^3 y - x)dx + (y^4 x^2 + 1)dy = 0$ is

(a) homogeneous;
(b) exact;
(c) Bernoulli;
(d) none of the above

Answer: (d)

Problem 2.[20 points] For the initial value problem

$$\frac{dy}{dx} = x - y, \quad y(1) = 4,$$

perform one step of the Improved Euler Method with step $h = .5$ (that is compute y_0, y_1 for $x_0 = 1, x_1 = 1.5$). Give all the details of your work.

Solution.

We have $f(x,y) = x - y$, $x_0 = 1, x_1 = 1.5$, $h = .5$ and $y_0 = .5$ Now $k_1 = f(x_0, y_0) = 1 - 4 = -3$ and $u_1 = y_0 + hk_1 = 4 - \frac{3}{2} = \frac{5}{2}$. Then $k_2 = f(x_1, u_1) = \frac{3}{2} - \frac{5}{2} = -1$. Finally,

$$y_1 = y_0 + h\frac{1}{2}(k_1 + k_2) = 4 + \frac{1}{2}\frac{1}{2}(-3 - 1) = 4 - 1 = 3.$$

Problem 3.[22 points] Find the general solution of the equation

$$y' + \frac{2y}{x} = \sqrt{x}$$

on the interval $x > 0$. Give all the details of your work.

Solution. Since this is a linear equation of first order, we start by computing the integrating factor:

$$\rho(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2.$$

After multiplying the main equation by $\rho(x) = x^2$ we get

$$\frac{d}{dx}(x^2 y) = x^{5/2}$$

$$x^2 y = \int x^{5/2}\, dx = \frac{2}{7}x^{7/2} + C$$

$$y = \frac{2}{7}x^{3/2} + Cx^{-2}.$$

Problem 4.[20 points] Find the general solution of the differential equation

$$(2y - 1)dx + (2x + 3y^2)dy = 0.$$

Give all the details of your work.

Solution.

Since $\frac{\partial}{\partial y}(2y - 1) = 2 = \frac{\partial}{\partial x}(2x + 3y^2)$, this is an exact equation. So its general solution is $F(x, y) = C$ where F is such that $\frac{\partial F}{\partial x} = 2y - 1$ and $\frac{\partial F}{\partial y} = 2x + 3y^2$.

From $\frac{\partial F}{\partial x} = 2y - 1$ we get $F = \int(2y - 1)dx = 2xy - x + g(y)$. Then $\frac{\partial F}{\partial y} = 2x + g'(y) = 2x + 3y^2$ and so $g'(y) = 3y^2$. This yields $g(y) = y^3$ and $F(x, y) = 2xy - x + g(y) = 2xy - x + y^3$.

Hence the general solution of the original equation is $2xy - x + y^3 = C$, where C is a constant.

Problem 5. [22points]

Solve the initial value problem

$$y'' - 10y' + 25y = 0, y(0) = 1, y'(0) = -3.$$

Give all the details of your work.

Solution. The characteristic equation is $r^2 - 10r + 25 = (r-5)^2 = 0$ which has a repeated real root $r_{1,2} = 5$. Therefore the general solution of $y'' - 10y' + 25y = 0$ is $y = c_1 e^{5x} + c_2 x e^{5x}$. This gives $y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$.

Substituting the initial conditions $y(0) = 1, y'(0) = -3$ we get $c_1 = 1$ and $5c_1 + c_2 = -3$, yielding $c_2 = -8$.

Therefore the solution of the original initial value problem is

$$y = e^{5x} - 8xe^{5x}.$$