Quiz 7 (Solution) October 21, 2005

Problem 1.

Find all the **positive** eigenvalues and the corresponding eigenfunctions of the following problem:

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y'(1) = 0. \end{cases}$$

Solution.

Let $\lambda > 0$. Then there is a unique $\alpha > 0$ such that $\lambda = \alpha^2$. The above system becomes

$$(*) \qquad \begin{cases} y'' + \alpha^2 y = 0 \\ y(0) = y'(1) = 0. \end{cases}$$

The equation $y'' + \alpha^2 y = 0$ has characteristic equation $r^2 + \alpha^2 = 0$ with two complex conjugate roots $r_{1,2} = \pm\alpha i$. Thus the general solution of $y'' + \alpha^2 y = 0$ is

$$y = A\cos\alpha x + B\sin\alpha x, \quad A, B \in \mathbb{R}.$$

By differentiating we get

$$y' = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Using $y(0) = 0$ we obtain $y(0) = A \cdot 1 + B \cdot 0 = 0$ and hence $A = 0$. Thus $y = B\sin\alpha x$ and $y' = B\alpha\cos\alpha x$. Using the condition $y'(1) = 0$ we get

$$B\alpha\cos\alpha = 0.$$

We know that $\cos t = 0$ if and only if $t = \frac{\pi}{2} + \pi n$ for some $n \in \mathbb{Z}$.

If $\alpha > 0$ and $\alpha \neq \frac{\pi}{2} + \pi n$ for some $n \in \mathbb{Z}$, then $\cos\alpha \neq 0$ and $B\alpha\cos\alpha = 0$ implies $B = 0$ and hence $y = 0$. In that case $\lambda = \alpha^2$ is not an eigenvalue.

If $\alpha > 0$ has the form $\alpha = \frac{\pi}{2} + \pi n = \frac{\pi(2n+1)}{2}$ for some $n \in \mathbb{Z}$ then $\cos\alpha = 0$. Thus $B\alpha\cos\alpha = 0$ for every $B \in \mathbb{R}$ and $y = B\sin\alpha x$ is a solution of the system (*) for every $B \in \mathbb{R}$. Therefore in this case $\lambda = \alpha^2$ is an eigenvalue with an eigenfunction $y = \sin\alpha x$.

Thus the positive eigenvalues are $\lambda_n = \frac{\pi^2(2n+1)^2}{4}$ with eigenfunctions $y_n = \sin\frac{\pi(2n+1)x}{2}$, where $n = 0, 1, 2, \ldots$.