

Problem 1.

Find the Fourier series of the 4-periodic function $f(t)$ such that

$$f(t) = |t|, \quad \text{for } -2 \leq t \leq 2.$$

Solution.

The half-period is $L = 2$. Note that the function $f(t)$ is even. Therefore $b_n = 0$ for all $n \geq 1$. Also, since $f(t)$ is even, we have for each $n \geq 0$

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos \frac{\pi n t}{L} dt = \frac{2}{L} \int_0^L f(t) \cos \frac{\pi n t}{L} dt = \frac{2}{2} \int_0^2 t \cos \frac{\pi n t}{2} dt.$$

Thus

$$a_0 = \int_0^2 t dt = \left[\frac{t^2}{2} \right]_0^2 = 2$$

and for $n \geq 1$

$$a_n = \int_0^2 t \cos \frac{\pi n t}{2} dt =$$

$$\begin{aligned} & \text{integration by parts with } u = t, \quad dv = \cos \frac{\pi n t}{2} dt, \quad v = \frac{2}{\pi n} \sin \frac{\pi n t}{2} \\ &= \left[\frac{2}{\pi n} t \sin \frac{\pi n t}{2} \right]_0^2 - \frac{2}{\pi n} \int_0^2 \sin \frac{\pi n t}{2} dt = 0 - \frac{2}{\pi n} \int_0^2 \sin \frac{\pi n t}{2} dt = \\ &= \frac{4}{\pi^2 n^2} \left[\cos \frac{\pi n t}{2} \right]_0^2 = \frac{4}{\pi^2 n^2} (\cos(\pi n) - 1) = \frac{4}{\pi^2 n^2} ((-1)^n - 1) = \\ &= \begin{cases} \frac{-8}{\pi^2 n^2}, & \text{for } n - \text{ odd} \\ 0, & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

Therefore the Fourier series of $f(t)$ is

$$f(t) \sim 1 - \frac{8}{\pi^2} \sum_{n-\text{ odd}} \frac{1}{n^2} \cos \frac{\pi n t}{2}.$$