Quiz 9 (Solutions) November 18, 2005

Problem 1.

Solve the following heat equation problem:

$$\begin{cases} u_t = 5u_{xx}, \qquad 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \\ u(x,0) = \sin(2x) + 10\sin(4x), \qquad 0 < x < \pi. \end{cases}$$

Solution.

This is a heat equation problem with zero endpoint temperatures. Here $k = 5$, $L = \pi$ and the initial temperature function is $f(x) = u(x,0) = 2\sin(2x) + 10\sin(4x)$ for $0 < x < \pi$. According to Theorem 1 of Ch. 9.5, this problem has the solution

$$u = \sum_{n=1}^{\infty} b_n \exp(-n^2\pi^2 kt/L^2) \sin\frac{\pi n x}{L} = \sum_{n=1}^{\infty} b_n \exp(-5n^2 t)\sin(nx),$$

where b_n are the Fourier Sine Series coefficients of $f(x)$. Since $\sin(2x)$ and $\sin(7x)$ are already odd and 2π-periodic functions, the odd extension of $f(x)$ is defined as $f_O(x) = 2\sin(2x) + 10\sin(4x)$ for every $x \in \mathbb{R}$. In this case the Fourier Sine Series coefficients can be "read-off" from the function $f(x)$ directly:

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \sin(2x) + 10\sin(4x), \qquad -\infty < x < \infty,$$

gives us $b_2 = 1$, $b_4 = 10$ and $b_n = 0$ for $n \neq 2, n \neq 4$. Therefore

$$u(x,t) = \exp(-5\cdot 4t)\sin(2x) + 10\exp(-5\cdot 16t)\sin(4x) =$$
$$= e^{-20t}\sin(2x) + 10e^{-80t}\sin(4x).$$