Math 417 Exam 1 (with solutions)

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Problem 1.[20 points]
(1) Let a, b be even integers. Suppose there exist integers s, t such that $sa + tb = 2$. Prove that $gcd(a, b) = 2$.
Solution. Let $d = gcd(a, b)$. Then $d|a$, $d|b$ and hence $d|(sa + tb)$, that is $d|2$. Thus $d \leq 2$. On the other hand, a abd b are even. Therefore 2 is a common divisor of a, b and hence $2 \leq gcd(a, b) = d$. Thus $2 \leq d \leq 2$ and so $d = 2$.
(2) Show that $\sqrt[3]{2}$ is irrational.
Solution. Suppose, on the contrary, that $\sqrt[3]{2}$ is rational. Then we can express $\sqrt[3]{2}$ as a fraction of two positive integers in lowers terms:

$$\sqrt[3]{2} = \frac{a}{b}, \text{ where } gcd(a, b) = 1, a > 0, b > 0.$$

Raising to the third power and cross-multiplying the above equation we get

$$2 = \frac{a^3}{b^3}, \quad 2b^3 = a^3.$$

Thus $2|a^3$. Since 2 is prime, Euclid's lemma implies that $2|a$. Thus $a = 2m$ for some integer $m > 0$. Therefore

$$2b^3 = (2m)^3 = 8m^3 \text{ and so } b^3 = 4m^3.$$

Thus $2|b^3$. By Euclid's lemma this again means that $2|b$. Thus $2|a$ and $2|b$ which contradicts the assumption that $(a, b) = 1$.
Problem 2.[20 points]
Prove that for every integer $n \geq 2$

$$n = \sum_{r=1}^{n} \binom{n}{r} r 5^{n-r} (-4)^{r-1}$$

Solution.

By the Binomial Theorem we have:

$$(5 + x)^n = \sum_{r=0}^{n} \binom{n}{r} 5^{n-r} x^r$$

Differentiating both sides along x we obtain:

$$n(5 + x)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r 5^{n-r} x^{r-1}$$

(Note that the term corresponding to $r = 0$ was constant and hence vanishes after differentiation).

Substituting $x = -4$ in the above formula we get:

$$n = n(5 - 4)^n = \sum_{r=1}^{n} \binom{n}{r} r 5^{n-r} (-4)^{r-1}.$$

Problem 3.[20 points]
(a) Find the general solution for the following system of congruence equations:

$$\begin{cases} x \equiv 2 \mod 6 \\ 3x \equiv 1 \mod 7 \end{cases} \tag{1}$$

Solution.
The fiirst equation is equivalent to saying that $x = 6k + 2$, where $k \in \mathbb{Z}$. Substituting this into the second equation we get

$$\begin{aligned} 18k + 6 &\equiv 1 \mod 7 \\ 18k &\equiv -5 \mod 7. \end{aligned}$$

We have $1 = 36 - 35 = 2 \cdot 18 - 5 \cdot 7$. Multiplying this by -5 we get $-5 = (-10) \cdot 18 + 25 \cdot 7$, so that $k_0 = -10$ satisfies $18k \equiv -5 \mod 7$. Substituting $k_0 = -10$ in $x = 6k + 2$ we get $x_0 = 6 \cdot (-10) + 2 = -58$ is a particular solution of (1). Therefore by the Chinese Remainder Theorem the general solution of (1) is $x \equiv -58 \mod 42$, that is $x \equiv -16 \mod 42$.

(b) Find the general solution for the following congruence equation:

$$6x \equiv 1 \mod 14$$

Solution.
If x is an integer such that $6x \equiv 1 \mod 14$ then $6x - 1 = 14k$ for some integer k. Since $2|14$, the number $14k$ is even. However, the number $6x$ is even and hence $6x-1$ is odd. Thus $6x-1 = 14k$ is both even and odd, which is a contradiction. Hence the congruence $6x \equiv 1 \mod 14$ has no solutions.

Problem 4.[20 points]
Let $\alpha = (123456789) \in S_n$, where $n \geq 9$.

(a) Prove that α cannot be represented as a product $\alpha = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7$ where γ_i are 4-cycles.
(b) Let $\alpha' = (12)(34) \in S_n$. Can α and α' be conjugate in S_n? Explain why.

Solution.
(a) Recall that if ω is an r-cycle then $sgn(\omega) = (-1)^{r-1}$.

Suppose that α admits a representation as in (a). Thus $sgn(\alpha) = (-1)^{9-1} = 1$ and $sgn(\gamma_i) = (-1)^{4-1} = -1$.

By the multiplicative property of sgn we have

$$1 = sgn(\sigma) = \prod_{i=1}^{7} sgn(\gamma_i) = (-1)^7 = -1,$$

yielding a contradiction.

(d) The permutations α and α' have different cycle structure since the complete factorization of α contains a 9-cycle while the the complete factorization of α' does not contain a 9-cycle. Hence α and α' cannot be conjugate in S_n.

Problem 5.[20 points]

Let $a_1 = 1$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

Prove that for every $n \geq 1$

$$a_n \leq a_{n+1}. \tag{!}$$

Solution.

We will prove this statement by induction.

Base of Induction.

For $n = 1$ we have $a_1 = 1 \leq \sqrt{3} = a_2$.

Inductive Step.

Suppose $n \geq 1$ and the statement (!) is known to hold for a_n. We need to deduce that $a_{n+1} \leq a_{n+2}$.

We have $a_{n+1} = \sqrt{2 + a_n} \geq 0$ and $a_{n+2} = \sqrt{2 + a_{n+1}} \geq 0$.

Therefore $a_{n+1} \leq a_{n+2}$ is equivalent to

$$a_{n+1}^2 \leq a_{n+2}^2 \iff$$
$$2 + a_n \leq 2 + a_{n+1} \iff a_n \leq a_{n+1}$$

where the last inequality holds by the Inductive Hypothesis.

This completes the Inductive Step.