Math 417 Exam 2 (SOLUTIONS)

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Problem 1. For each of the following decides if it is a group. Justify your answers.
(1) $(\mathbb{R}, \cdot)$, where $\cdot$ is the standard multiplication of the real numbers.
(2) $(\mathbb{Z}, *)$ where by definition $a * b := 0$ for every $a, b \in \mathbb{Z}$.
(3) $(Y_n, \cdot)$ where $n \geq 2$ and Y_n is the set of all odd permutations in S_n and where $\cdot$ is the standard multiplication of permutations.

Solution.

(1) This is not a group since $0 \in \mathbb{R}$ has no inverse.

(2) This is not a group since there is no unit element. Indeed, for any $e \in \mathbb{Z}$ we have $2 * e = 0 \neq 2$.

(3) This is not a group since $\cdot$ is not a binary operation on Y_n. Indeed, $(1\ 2) \in Y_n$ but $(1\ 2) \cdot (1\ 2) = 1 \notin Y_n$.

Problem 2.

For each of the following quotient groups find a (non-quotient) group isomorphic to it. Justify your answers.
(a) $GL(n, \mathbb{C})/SL(n, \mathbb{C})$;
(b) $\mathbb{R}^\times/\mathbb{R}_{>0}$ where $\mathbb{R}_{>0}$ is the multiplicative group of positive real numbers.
(c) $\mathbb{C}^\times/\mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Solution.

In all three cases we will apply the First Isomorphism Theorem.

(a) The map $\det : GL(n, \mathbb{C}) \to \mathbb{C}^\times$ is an onto homomorphism with $ker(\det) = SL(n, \mathbb{C})$. Therefore $GL(n, \mathbb{C})/SL(n, \mathbb{C}) \cong \mathbb{C}^\times$.

(b) The map $f : \mathbb{R}^\times \to \{-1, 1\} = C_2$ defined as $f(x) = \frac{x}{|x|}$, is an onto homomorphism with $ker(F) = \mathbb{R}_{>0}$. Therefore $\mathbb{R}^\times/\mathbb{R}_{>0} \cong C_2$.

(c) The map $\phi : \mathbb{C}^\times \to \mathbb{R}^\times$ defined as $\phi(z) = |z|$, is an onto homomorphism with $ker(\phi) = \mathbb{S}^1$. Therefore $\mathbb{C}^\times/\mathbb{S}^1 \cong \mathbb{R}^\times$.

Problem 3.

For each of the following pairs of groups prove that they are not isomorphic.
(a) $\mathbb{Z} \times \mathbb{Z}$ and $SL(2, \mathbb{Z})$;
(b) $\mathbb{I}_4$ and $I_2 \times I_2$;
(c) $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$.

Solution.

(a) The group $SL(2, \mathbb{Z})$ is non-abelian while $\mathbb{Z} \times \mathbb{Z}$. Therefore $\mathbb{Z} \times \mathbb{Z} \ncong SL(2, \mathbb{Z})$.

(b) The group $\mathbb{I}_4$ has an element of order 4 while $I_2 \times I_2$ has no elements of order 4. Therefore $\mathbb{I}_4 \ncong I_2 \times I_2$.

(c) The group $\mathbb{Z}$ is cyclic while $\mathbb{Z} \times \mathbb{Z}$ is not cyclic. Therefore $\mathbb{Z} \not\cong \mathbb{Z} \times \mathbb{Z}$.

Problem 4.

Let G be a finite group such that for every element $y \in G$ there is $x \in G$ such that $y = x^3$.

Prove that for each $y \in G$ there exists a unique $x \in G$ such that $y = x^3$.

Solution.

Consider the function $f : G \to G$ defined as $f(x) = x^3$ for every $x \in G$. By assumption this function is surjective. Since G is a finite set, it follows that f is injective. Therefore for each $y \in G$ there exists a unique $x \in G$ such that $y = x^3$.

Problem 5.

Let G be a group and let $H \leq G$ be a subgroup. The *normalizer* $N_G(H)$ is defined as

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

(i) Prove that $N_G(H) \leq G$ and that H is a normal subgroup of $N_G(H)$.

(ii) Prove that $[G : N_G(H)]$ is equal to the number of subgroups of G conjugate to H.

Solution.

Consider the action of G in the set X of all subgroups of G defined as

$$g \cdot K := gKg^{-1}, \quad \text{where } , g \in G, K \leq G.$$

Then $N_G(H)$ is the stabilizer G_H of H for this action. Therefore $N_G(H) \leq G$. The orbit of H consists of all subgroups of G conjugate to H. Therefore by the counting formula $[G : N_G(H)] = [G : G_H]$ is equal to the number of subgroups of G conjugate to H.

To see that H is a normal subgroup of $N_G(H)$, observe first that H is contained in $N_G(H)$. Indeed, if $h \in H$ then $hHh^{-1} = H$ and hence $H \leq N_G(H)$. By definition of $N_G(H)$ if $g \in N_G(H)$ then $gHg^{-1} = H$. Therefore H is a normal subgroup of $N_G(H)$.