H/wk 6 (Selected Solutions)

2.49

If H and K are finite subgroups of G and if $|H|$ and $|K|$ are relatively prime, prove that $|H \cap K| = \{1\}$.

Solution.

We have $H \cap K \leq H$ and hence by Lagrange's theorem $|H \cap K|$ divides $|H|$. Similarly, $H \cap K \leq K$ and hence by Lagrange's theorem $|H \cap K|$ divides $|K|$. Thus $|H \cap K|$ is a common divisor of $|H|$ and $|K|$. Since $gcd(|H|, |K|) = 1$, it follows that $|H \cap K| = 1$ and hence $|H \cap K| = \{1\}$.

2.50 Prove that if G is an infinite group then G contains infinitely many subgroups.

Solution.

We will show that G contains infinitely many distinct cyclic subgroups.

There are two cases to consider.

Case 1. There is some element $g \in G$ such that g has infinite order.

Put $H_n := \langle g^n \rangle \leq G$ for $n = 1, 2, 3, \ldots$. Each H_n is an infinite cyclic group. Then $H_n \neq H_m$ for $n \neq m$. Indeed, if $n < m$ then $g^n \in H_n$ but $g^n \notin H_m$ and hence $H_n \neq H_m$.

Thus G contains infinitely many distinct cyclic subgroups, as claimed.

Case 2. Every element $g \in G$ has finite order.

Suppose that our claim fails and that G contains only finitely many cyclic subgroups. Then

$$\{\langle g \rangle | g \in G\} = \{\langle g_1 \rangle, \ldots, \langle g_n \rangle\}$$

for some $g_1, \ldots, g_n \in G$.

We have

$$G = \cup_{g \in G} \langle g \rangle = \cup_{i=1}^{n} \langle g_i \rangle.$$

By assumption each g_i has finite order and so each subgroup $\langle g_i \rangle$. Therefore the set $G = \cup_{i=1}^{n} \langle g_i \rangle$ is finite, which contradicts our assumption that G is and infinite group.

2.54(i)

Let $(S_i)_{i \in I}$ be a family of subgroups of a group G and let $(x_i)_{i \in I}$ be a family of elements of G. Prove that the intersection $\cap_{i \in I} x_i S_i$ is either empty or is a left coset of the subgroup $\cap_{i \in I} S_i$.

Solution.

Put $H := \cap_{i \in I} S_i$. Suppose that $X := \cap_{i \in I} x_i S_i$ is nonempty.

First we will show that for any $a, b \in X$ we have $aH = bH$, that is, $a^{-1}b \in H$. Indeed, let $a, b \in X$. Then for every $i \in I$ there are $s_i, s_i' \in S_i$ such that $a = x_i s_i$ and $b = x_i s_i'$. Therefore

$$a^{-1}b = s_i^{-1} x_i^{-1} x_i s_i' = s_i^{-1} s_i' \in S_i$$

for every $i \in I$. Hence $a^{-1}b \in \cap_{i \in I} S_i = H$, as required.

Since X is nonempty, choose some $a \in X$. The above statement implies that $X \subseteq aH$. Now let $c \in aH$ be arbitrary. Thus there is some $h \in H$ such that $c = ah$. We have $a \in X = \cap_{i \in I} x_i S_i$ and therefore $a \in x_i S_i$ for each $i \in I$. Since $h \in H \leq S_i$, it follows that $h \in S_i$ and $ah \in aS_i$ for every $i \in I$. Therefore

$$c = ah \in \cap_{i \in I} x_i S_i = X.$$

Since $c \in aH$ was arbitrary, this implies that $aH \subseteq X$. Together with the inclusion $X \subseteq aH$, established above, it follows that $X = aH$, as claimed.

2.54(ii)

Suppose a group G is the union of finitely many cosets $G = \cup_{i=1}^{n} x_i S_i$ of subgroups $S_i \leq G$. Prove that at least one of the subgroups S_i has finite index in G.

Solution.

We will prove this statement by induction on

$$m = |\{S_1, \ldots, S_n\}|,$$

the number of distinct subgroups among $S_1, \ldots, S_n$.

That is, we will prove that if a group G can be represented as the union of finitely many cosets of m distinct subgroups of G, then at least one of these subgroups has finite index in G.

Base of Induction. Let $m = 1$. Then there is $S \leq G$ such that $S_i = S$ for $i = 1, \ldots n$. Therefore

$$G = \cup_{i=1}^{n} x_i S.$$

By definition of the index $[G : S]$ it follows that $[G : S] \leq n < \infty$, as required.

Inductive Step.

Suppose that the statement is known for $m \geq 1$ and assume

$$|\{S_1, \ldots, S_n\}| = m + 1.$$

Let $\{S_1, \ldots, S_n\} = \{H_1, \ldots, H_m, H_{m+1}\}$ where $H_i \neq H_j$ for $i \neq j$. If $[G : H_{m+1}] < \infty$, then there is nothing to prove.

Suppose now that $[G : H_{m+1}] = \infty$. We re-express the union $G = \cup_{i=1}^n x_i S_i$. as

$$G = \big(\cup_{i=1}^p \, x_i S_i \big) \bigcup \big(\cup_{j=1}^k \, y_j H_{m+1} \big),$$

where $p + k = n$ and where $S_i \in \{H_1, \ldots, H_m\}$ for $i = 1, \ldots, p$. Since $[G : H_{m+1}] = \infty$, there is $a \in G$ such that $aH_{m+1} \neq y_j H_{m+1}$ for $j = 1, \ldots, k$. Hence $aH_{m+1} \cap y_j H_{m+1} = \emptyset$ for $j = 1, \ldots, k$. Since $aH_{m+1} \subset G$, it follows that

$$aH_{m+1} \subseteq \cup_{i=1}^p x_i S_i.$$

Therefore

$$H_{m+1} \subseteq \cup_{i=1}^p a^{-1} x_i S_i$$

and

$$y_j H_{m+1} \subseteq \cup_{i=1}^p y_j a^{-1} x_i S_i.$$

Thus

$$G = \cup_{i=1}^p x_i S_i \bigcup \cup_{j=1}^k \cup_{i=1}^p \, y_j a^{-1} x_i S_i.$$

So G is the union of finitely many cosets of subgroups $H_1, \ldots, H_m$. Hence by the inductive hypothesis there is some $i \leq m$ such that $[G : H_i] < \infty$, as required.