H/wk 7 (Selected Solutions)

2.60

(i) Prove that every group G with $|G| < 6$ is abelian.

(ii) Find two nonisomorphic groups of order 6.

Solution.

(i) If $|G| < 6$ then $|G| \in \{1, 2, 3, 4, 5\}$.

First we show that every group G, where $|G| = p$ is a prime, is abelian. Indeed, let $|G| = p > 1$ be a prime. Take any $g \in G$ such that $g \neq 1$. Then $ord(g) > 1$. We know that $ord(g)$ divides $|G| = p$. Since p is a prime, it follows that $ord(g) = p$. Hence $p = ord(g) = |\langle g \rangle| = |G|$. Therefore $G = \langle g \rangle$, so that G is cyclic and therefore abelian.

This shows that if $|G| \in \{2, 3, 5\}$ then G is abelian. If $|G| = 1$ then G is the trivial group and hence it is abelian. It remains to consider the case when $|G| = 4$.

Suppose first that G has an element g of order 4. Then $G = \langle g \rangle$ is cyclic of order 4 and therefore G is abelian.

Suppose now that G has no elements of order 4, so that every element of G has order smaller than 4. The order of every element of G must be a divisor of $|G| = 4$. Hence every nontrivial element of G has order 2. This means that for every $g \in G$ we have $g^2 = 1$. By the result of problem 2.38 it follows that G is abelian.

(ii) The groups S_3 and C_6 both have order 6. The group C_6 is abelian while S_3 is not. Therefore C_6 is not isomorphic to S_3.

2.64

(i) Find a subgroup $H \leq S_4$ such that $H \cong \mathbf{V}$ but $H \neq \mathbf{V}$.

(ii) Show that H is not normal in S_4.

Solution.

(i) The group $\mathbf{V}$ has the properties that $|\mathbf{V}| = 4$, that $x^2 = 1$ for every $x \in \mathbf{V}$ and that the product of any two distinct nontrivial elements of $\mathbf{V}$ is equal to the remaining nontrivial element. Moreover, these properties completely determine the multiplication table for $\mathbf{V}$ and hence any group with these properties is isomorphic to $\mathbf{V}$.

Consider $H = \{1, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\} \subset S_4$. It is easy to see that $H \leq S_4$. Moreover, $|H| = 4$, the square of every element of H is trivial and the product of any two distinct nontrivial elements of H is equal to the remaining nontrivial element. Therefore $H \cong \mathbf{V}$.

Any bijection between H and $\mathbf{V}$ which takes $1 \in H$ to $1 \in \mathbb{V}$ is an isomorphism between H and $\mathbf{V}$.

(ii) We have $(1\ 2) \in H$ but

$$(1\ 2\ 3\ 4)(1\ 2)(1\ 2\ 3\ 4)^{-1} = (1\ 3) \notin H.$$

Therefore H is not normal in G.