H/wk 8 (Selected Solutions)

2.85

Prove that every group $U(\mathbb{I}_9) \cong \mathbb{I}_6$ and that $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.

Solution.

By Lemma 2.106 $U(\mathbb{I}_9) = \{[1]_9, [2]_9, [4]_9, [5]_9, [7]_9, [8]_9\}$. Thus $|U(\mathbb{I}_9)| = 6$. To prove that $U(\mathbb{I}_9) \cong \mathbb{I}_6$ it suffices to show that $U(\mathbb{I}_9)$ is a cyclic group of order 6. To see that $U(\mathbb{I}_9)$ is a cyclic group of order 6 it is enough to find an element of order 6 in $U(\mathbb{I}_9)$. Take $x = [2]_9$. Then $x^2 = [4]_9$, $x^3 = [8]_9$, $x^4 = [16]_9 = [7]_9$, $x^5 = [32]_9 = [5]_9$, $x^6 = [64]_9 = [1]_9$. Thus x has order 6 in $U(\mathbb{I}_9)$ and therefore $U(\mathbb{I}_9) \cong \mathbb{I}_6$, as required.

By Lemma 2.106

$$U(\mathbb{I}_{15}) = \{[1]_{15}, [2]_{15}, [4]_{15}, [7]_{15}, [8]_{15}, [11]_{15}, [13]_{15}, [14]_{15}\}.$$

To show that $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$ it suffices to find subgroups $H, K \lhd U(\mathbb{I}_{15})$ such that H is cyclic of order 4, K is cyclic of order 2, such that $HK = U(\mathbb{I}_{15})$ and $H \cap K = \{1\}$.

Take $H = \langle [2]_{15} \rangle \leq U(\mathbb{I}_{15})$ and $K = \langle [14]_9 \rangle \leq U(\mathbb{I}_{15})$. Note that $[14]_{15} = [-1]_{15}$ and therefore $[14^2]_{15} = [(-1)^2]_{15} = [1]_{15}$, so that $[14]_{15}$ has order 2 in $U(\mathbb{I}_{15})$ and $K = \{[1]_{15}, [14]_{15}\}$ is cyclic of order 2.

Similarly we check that $H = \langle [2]_{15} \rangle = \{[1]_{15}, [2]_{15}, [4]_{15}, [8]_{15}\}$ is cyclic of order 4. Thus $H \cap K = \{[1]_{15}\}$. Since $U(\mathbb{I}_{15})$ is abelian, both H and K are normal on $U(\mathbb{I}_{15})$. We have $[7]_{15} = [-8]_{15} = [8]_{15}[-1]_{15} \in HK$, $[11]_{15} = [-4]_{15} = [4]_{15}[-1]_{15} \in HK$ and $[13]_{15} = [-2]_{15} = [2]_{15}[-1]_{15} \in HK$. Thus $HK = U(\mathbb{I}_{15})$. All the required conditions have been verified and hence $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.

2.93

(i) Prove that $\mathbf{Q}/Z(\mathbf{Q}) \cong \mathbf{V}$

(ii) Prove that $\mathbf{Q}$ has no subgroup isomorphic to $\mathbf{V}$.

Solution.

(i) Recall that

$$\mathbf{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$$

where $i^2 = j^2 = k^2 = -1$ and $ij = k, ji = -k$, $ki = j, ik = -j$, $jk = i$, $kj = -i$.

It follows that elements $1, -1$ are the only central elements of $\mathbf{V}$ so that $Z(\mathbf{V}) = \{1, -1\}$. Hence $|\mathbf{Q}/Z(\mathbf{Q})| = 8/2 = 4$. Put $H = Z(\mathbf{Q}) = \{\pm 1\}$.

We see that

$$\mathbf{Q}/H = \{H, iH, kH, jH\}$$

and that in $\mathbf{Q}/H$

$$(iH)^2 = (kH)^2 = (jH)^2 = H, iHjH = jHiH = kH,$$
$$kHiH = iHkH = jH, jHkH = kHjH = iH.$$

Thus $\mathbf{Q}/H$ is a group of order 4 where every nontrivial element has order 2 and where the product of any two distinct nontrivial elements is equal to the remaining nontrivial element. Therefore $\mathbf{Q}/H \cong \mathbf{V}$, as required.

(ii) From the definition of $\mathbf{Q}$ we see that the elements $\pm i, \pm k, \pm j$ have order 4 and the only element of order 2 in $\mathbf{Q}$ is -1. However the group $\mathbf{V}$ has three elements of order 2. Therefore $\mathbf{Q}$ does not have a subgroup isomorphic to $\mathbf{V}$.