Quiz 10 (solutions); Friday, April 22, 2005

For each of the following determine whether it is a field. If not, explain why not.

1. $\mathbb{Z}$;
2. $\{a + bi | a, b \in \mathbb{Q}\}$;
3. $\mathbb{C}[x]$.

Solution.

(1) $\mathbb{Z}$ is not a field since $2 \in \mathbb{Z}$ has no multiplicative inverse in $\mathbb{Z}$.

(2) $S = \{a + bi | a, b \in \mathbb{Q}\}$ is a field since it is a subfield of $\mathbb{C}$. Note that if $z = a + bi \in S, z \neq 0$ then

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \in S,$$

since for $a, b \in \mathbb{Q}$ with $a^2 + b^2 \neq 0$ we have $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \in \mathbb{Q}$.

(3) $\mathbb{C}[x]$ is not a field since $x \in \mathbb{C}[x]$ has no multiplicative inverse in $\mathbb{C}[x]$. Indeed, if $f(x) \in \mathbb{C}[x]$, $f \neq 0$, then

$$deg(xf) = deg(x) + deg(f) = 1 + deg(f) \geq 1 > 0,$$

and hence $xf \neq 1 \in \mathbb{C}[x]$.