Quiz 4 (Solutions); Friday, February 25

For each of the following determine if it is a group. If it is not a group, explain why not.

(a) $(SL_n(\mathbb{R}), \cdot)$, where $\cdot$ is the matrix multiplication and

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}.$$

(b) $(\mathbb{R}, \cdot)$, where $\cdot$ is the standard multiplication of real numbers

(c) $(\mathbb{Z}, \wedge)$, where by definition $x \wedge y = y$ for every $x, y \in \mathbb{Z}$.

(d) $(X, \cdot)$ where $X = \{z \in \mathbb{C} : |z| = 2^n \text{ for some } n \in \mathbb{Z}\}$ and where $\cdot$ is the standard complex multiplication.

Solution.

(a) Yes, this is a group.

(b) No, this is not a group. The operation $\cdot$ is associative and $1 \in \mathbb{R}$ is a unit element. However, $0 \in \mathbb{R}$ does not have an inverse, since there does not exist $x \in \mathbb{R}$ such that $x \cdot 0 = 1$.

(c) No, this is not a group. The operation $\wedge$ is associative, but there is no unit element. Indeed, suppose there exists $e \in \mathbb{Z}$ such that $x \wedge e = e \wedge x = x$ for every $x \in \mathbb{Z}$. Choose $x \in \mathbb{Z}$ such that $x \neq e$, for example, $x = e + 1$. Then by definition of $\wedge$, we have $x \wedge e = e \neq x$, a contradiction.

(d) Yes, this is a group.