

Quiz 6 (Solutions); Friday, March 11, 2005

For each of the following maps determine if it is a homomorphism. If it is not, explain why not.

(a) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x, y) = x + 2y$ for every $(x, y) \in \mathbb{R}^2$. Here both \mathbb{R}^2 and \mathbb{R} are considered as additive groups.

(b) $f : S_n \rightarrow S_n$ where $n \geq 3$ and $f(\alpha) = (12)\alpha$ for every $\alpha \in S_n$.

(c) $f : GL(2, \mathbb{R}) \rightarrow \mathbb{R}^\times$ defined as $f(A) = [\det(A)]^2$ for every $A \in GL(2, \mathbb{R})$. Here the group operation on \mathbb{R}^\times is the standard multiplication of real numbers.

(d) $f : GL(2, \mathbb{R}) \rightarrow GL(2, \mathbb{R})$ defined as $f(A) = A^2$ for every $A \in GL(2, \mathbb{R})$.

Solution.

(a) Yes, this is a homomorphism.

Indeed,

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = x_1 + x_2 + 2(y_1 + y_2) = \\ &= (x_1 + 2y_1) + (x_2 + 2y_2) = f(x_1, y_1) + f(x_2, y_2) \end{aligned}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(b) No, this is not a homomorphism since $f(1) = (1\ 2) \neq 1$.

(c) Yes, this is a homomorphism.

(d) No, this is not a homomorphism.

Indeed,

$$\begin{aligned} f \text{ is a homomorphism} &\iff \\ f(AB) = f(A)f(B) \text{ for every } A, B \in GL(2, \mathbb{R}) &\iff \\ ABAB = A^2B^2 \text{ for every } A, B \in GL(2, \mathbb{R}) &\iff \\ BAB = AB^2 \text{ for every } A, B \in GL(2, \mathbb{R}) &\iff \\ BA = AB \text{ for every } A, B \in GL(2, \mathbb{R}). \end{aligned}$$

The last statement does not hold since $GL(2, \mathbb{R})$ is not abelian.

Note that the function f does satisfy the properties $f(I_2) = I_2$, where I_2 is the 2×2 identity matrix, and $f(A^{-1}) = [f(A)]^{-1}$ for every $A \in GL(2, \mathbb{R})$.