Extra Credit Problem Set 5 (Solutions); Due Friday, April 25.

1.

Let R be a commutative ring with 1. Let I be an ideal in $R[x]$ such that the lowest degree of a nonzero polynomial in I is $n \geq 1$ and such that R contains a monic (that is, with leading coefficient 1) polynomial of degree n. Prove that I is a principal ideal.

Solution.

The argument is essentially the same as for proving that for a field F all ideals in $F[x]$ are principal.

Lemma. Let R be a commutative ring with 1 and let $f(x) \in R[x]$ be a monic polynomial of $deg(f) \geq 1$. Then for any $g \in R[x]$ there exist $q, r \in R[x]$ such that $g = qf + r$ and such that either $deg(r) < deg(f)$ or $r = 0$.

Proof of Lemma. We prove this by induction on $k = deg(g)$. If $deg(g) = 0$, then $g = 0 \cdot f + g$ and the statement holds.

Suppose now $deg(g) = k > 0$ and that the lemma has been proved for all polynomials of degree $< k$. If $deg(f) > deg(g)$ then we have $g = 0 \cdot f + g$ and the statement holds with $q = 0$, $r = g$. Suppose now that $n = deg(f) \leq deg(g) = k$. Let b_k be the leading coefficient of g. Then for $h = g - b_k x^{k-n} f$ has the coefficient at x^m equal to 0 since f is monic. Thus $deg(h) < deg(g) = k$. By the inductive hypothesis applied to h there exist $q_1, r \in R[x]$ such that $h = q_1 f + r_1$ where $deg(r) < deg(f)$ or $r = 0$. Hence $g = (b_k x^{k-n} + q_1)f + r$ and the statement of the Lemma is established. □

Now I be an ideal in $R[x]$ such that the lowest degree of a nonzero polynomial in I is $n \geq 1$ and such that R contains a monic polynomial $f(x)$ of degree n. We claim that $\langle f \rangle = I$. Since $f \in I$, it is clear that $\langle f \rangle \subseteq I$.

Let $g \in I$ be arbitrary. Since f is monic, by Lemma there exist $q, r \in R[x]$ such that $g = qf + r$ and such that either $deg(r) < deg(f)$ or $r = 0$. We claim that $r = 0$. If not then $r \neq 0$ satisfies $deg(r) < deg(f) = n$. We have

$$r = g - qf \in I, \qquad \text{since } g, f \in I \lhd R[x].$$

Thus r is a nonzero polynomial in I of degree $deg(r) < deg(f) = n$. This contradicts the choice of f. Therefore $r = 0$ so that $q = qf \in \langle f \rangle$. Since $g \in I$ was arbitrary, it follows that $I \subseteq \langle f \rangle$. Therefore $I = \langle f \rangle$, as required.

Alternative Solution.

Let I be an ideal in $R[x]$ such that the lowest degree of a nonzero polynomial in I is $n \geq 1$ and such that R contains a monic polynomial $f(x)$ of degree n. We claim that $\langle f \rangle = I$.

We claim that $I = \langle f \rangle$. Since $f \in I$, is is obvious that $\langle f \rangle \subseteq I$.

We need to show that $I \subseteq \langle f \rangle$. Suppose that this is not true. Then there exists some nonzero polynomial which belongs to I but does not belong to $\langle f \rangle$. Among all nonzero polynomials that belong to I but do not belong to $\langle f \rangle$, choose a polynomial $g(x)$ of minimal degree.

Note that $k = deg(g) \geq deg(f) = n \geq 1$ by the choice of f. (Indeed, if $deg(g) < deg(f)$ then, since $g \neq 0$, $g \in I$, it would follow that I contains a nonzero polynomial of degree less than n, contrary to the choice of f).

Let $b_k \in R$, $b_k \neq 0$ be the leading coefficient of g. Put $h = g - b_k x^{k-n} f$. Then, since f is monic, the coefficient at x^k in h is 0, so that $deg(h) < deg(g)$. Since $f, g \in I$, it follows that $h \in I$. If $h \in \langle f \rangle$ then $g = b_k x^{k-n} f + h \in \langle f \rangle$ contrary to the choice of g. Hence $h \notin \langle f \rangle$ and, in particular, $h \neq 0$.

Thus $h \neq 0$, $h \in I$, $h \notin \langle f \rangle$ and $deg(h) < deg(g)$. This contradicts the choice of g.

Therefore $I \subseteq \langle f \rangle$ and so $I = \langle f \rangle$, as required.

2. Prove that for every $n \geq 1$ the ring $M_n(\mathbb{R})$ has no proper ideals. That is, prove that if $I \lhd M_2(\mathbb{R})$ then either $I = 0$ or $I = M_2(\mathbb{R})$.

Hint. It is enough to prove that if $A \in M_n(\mathbb{R})$ is a nonzero matrix then the principal ideal $\langle A \rangle$ is equal to $M_n(\mathbb{R})$. Think about the effect of multiplying on (the left or on the right) a matrix in $M_n(\mathbb{R})$ by an elementary matrix $E_{k,l}$ (where $(E_{k,l})_{i,j} = 1$ if $k = i, j = l$ and $(E_{k,l})_{i,j} = 0$ otherwise).

Solution.

Let $1 \leq k, l \leq n$, where $k, l \in \mathbb{Z}$. A direct computation shows that if $A \in M_n(\mathbb{R})$ is an arbitrary matrix then $E_{kl}A$ is the $n \times n$ matrix in whose k-th row we have the l-th row of A and where all the other rows of $E_{kl}A$ are filled with zeros. Similarly, AE_{kl} is the $n \times n$ matrix whose l-th column is the k-th column of A and where all the other columns of AE_{kl} are filled with zeros. It follows that for any $A \in M_n(\mathbb{R})$ and for any $1 \leq i, j, k, l \leq n$ the matrix $E_{il}AE_{kj}$ is the $n \times n$ matrix whose entry in the position ij is a_{lk} and all of whose other entries are zeros.

Now let $I \lhd M_n(\mathbb{R})$ be a nonzero ideal. We need to prove that $I = M_n(\mathbb{R})$.

Choose a nonzero matrix $A \in I$. Then for some l, k we have $a_{lk} \neq 0$.

Let $1 \leq i, j \leq n$ be arbitrary. Then the matrix $B = E_{il}AE_{kj}$ satisfies

$b_{ij} = a_{lk}$ and $b_{st} = 0$ for $(s,t) \neq (i,j)$, that is $B = E_{il}AE_{kj} = a_{lk}E_{ij}$. Since I is an ideal in $M(n,\mathbb{R})$, we have $B = a_{lk}E_{ij} \in I$.

Let $I_n \in M_n(\mathbb{R})$ be the identity matrix. Then $\frac{1}{a_{lk}}I_n$ is the diagonal matrix where all the diagonal entries are equal to $\frac{1}{a_{lk}}$. It is easy to see that $(\frac{1}{a_{lk}}I_n)(a_{lk}E_{ij}) = E_{ij}$. Since $a_{lk}E_{ij} \in I$ it follows that $E_{ij} \in I$ for every $1 \leq i,j \leq n$. Hence $I_n = \sum_i E_{ii} \in I$. Therefore for every matrix $C \in M_n(\mathbb{R})$ we have $C \cdot I_n = C \in M_n(\mathbb{R})$. Thus $I = M_n(\mathbb{R})$, as required.