Quiz 7 (Solutions); Friday, March 28, 2008

Problem 1.

For each of the following objects determine whether or not it is a ring. Justify your answers.

1. The set $\mathbb{Z}_+ = \{a \in \mathbb{Z} : a \geq 0\}$, under the usual operations of addition and multiplication of real numbers.
2. The set $GL(2, \mathbb{R})$, under matrix addition and multiplication.
3. The set
$$S = \{\frac{m}{2^n} \,|\, m, n \in \mathbb{Z}, n \geq 0\}$$
under the usual operations of addition and multiplication of real numbers.

Solution.

(a) This is not a ring since $(\mathbb{Z}_+, +)$ is not an abelian group: nonzero elements of $\mathbb{Z}_+$ do not have additive inverses in $\mathbb{Z}_+$.

(b) This is not a ring since matrix addition is not a binary operation on $GL(2, \mathbb{R})$, and hence $(GL(2, \mathbb{R}), +)$ is not an abelian group. For example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathbb{R})$ but
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin GL(2, \mathbb{R}).$$

(c) The set S is a subring of $\mathbb{Q}$ and hence S is a ring with respect to the usual operations of addition and multiplication of real numbers.

To show that S is a subring of $\mathbb{Q}$, we will apply the subring test.

Suppose $a = \frac{m}{2^n}, b = \frac{k}{2^l} \in S$ where $m, n, k, l \in \mathbb{Z}$ and $n, l \geq 0$.

Then
$$a - b = \frac{m}{2^n} - \frac{k}{2^l} = \frac{m2^l - k2^n}{2^{n+l}} \in S$$
and
$$ab = \frac{m}{2^n} \cdot \frac{k}{2^l} = \frac{mk}{2^{n+l}} \in S,$$
as required.