Math 427 Section E Exam 1 (Solutions)

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Problem 1.

Prove that every finitely generated subgroup of $(\mathbb{Q},+)$ is cyclic.

Solution

Let $H = \langle g_1, \dots, g_n\rangle \le \mathbb{Q}$.

We need to prove that H is cyclic. We may assume that $g_i \ne 0$ for $i = 1, \dots, n$.

Write $g_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{Z}$ are nonzero integers.

Since $\mathbb{Z}$ is abelian,

$$H = \{\sum_{i=1}^{n} s_i \frac{a_i}{b_i} | s_i \in \mathbb{Z}\}.$$

Let $g \in H$ be an arbitrary element. Thus $g = \sum_{i=1}^{n} s_i \frac{a_i}{b_i}$ for some $s_1, \dots, s_n \in \mathbb{Z}$.

Therefore

$$g = \frac{s_1 a_1 b_2 \dots b_n + s_2 a_2 b_1 b_3 \dots b_n + \dots + s_n a_n b_1 \dots b_{n-1}}{b_1 \dots b_n} \in \langle \frac{1}{b_1 \dots b_n}\rangle \le \mathbb{Q}.$$

Thus $H \le \langle \frac{1}{b_1 \dots b_n}\rangle \le \mathbb{Q}$.

The group $\langle \frac{1}{b_1 \dots b_n}\rangle$ is infinite cyclic and therefore, by Proposition 2.4 from the book, all of its subgroups are cyclic. Hence H is cyclic as well.

Problem 2.

A group G is called *directly indecomposable* if there do not exist nontrivial group H, K such that $G \cong H \times K$.

(a) Prove that the groups S_3 and $\mathbb{Z}$ are directly indecomposable.

(b) Prove that $(\mathbb{Q},+)$ is directly indecomposable.

Solution

(a) Suppose $S_3 \cong H \times K$, where H, K are nontrivial. Then $6 = |G| = |H| \cdot |K|$ and hence either $|H| = 2$, $|K| = 3$ or $|H| = 3, |K| = 2$.

We know that every group of order two is cyclic and every group of order three is cyclic. Hence a direct product of two such groups is abelian. However, S_3 is not abelian, yielding a contradiction.

In $\mathbb{Z}$ the intersection of any two nontrivial subgroups is nontrivial and hence $\mathbb{Z} \ncong H \times K$ where H, K are nontrivial.

(b) Suppose that $\mathbb{Q} \cong H \times K$ where H, K are nontrivial groups. Let $a \in H, b \in K$ be nontrivial elements. Then $\langle (a, 1), (1, b)\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

The group $\mathbb{Z} \times \mathbb{Z}$ is not cyclic while by Problem 1 every finitely generated subgroup of $\mathbb{Q}$ is cyclic, yielding a contradiction.

Problem 3.

Let A, B be groups and let $\phi : B \to Aut(A)$ be a homomorphism. Denote $\phi_b := \phi(b) \in Aut(A)$ for $b \in B$.

Define $\cdot$ on the set $A \times B$ as follows:

$$(a, b) \cdot (a', b') := (a\phi_b(a'), bb'), \qquad \text{where } a, a' \in A,\ b, b, \in B.$$

(1) Prove that the set $A \times B$, together with the operation $\cdot$ defined above, is a group.

[This group is called the *semi-direct product* of A and B along ϕ and is denoted $A \rtimes_\phi B$.]

(2) Let G be a group and $A, B, \leq G$ be subgroups of G such that A is normal in G, such that $AB = G$ and such that $A \cap B = \{1\}$. Prove that $G \cong A \rtimes_\phi B$ for some $\phi : B \to Aut(A)$.

Solution

(1) Let $a, a', a'' \in A,\ b, b', b'' \in B$.

Then

$$((a, b)(a', b'))(a'', b'') = (a\phi_b(a'), bb')(a'', b'') = (a\phi_b(a')\phi_{bb'}(a''), bb'b'')$$

and

$$(a, b)((a', b')(a'', b'')) = (a, b)(a'\phi_{b'}(a''), b'b'') = (a\phi_b[a'\phi_{b'}(a'')], bb'b'') =$$
$$(a\phi_b(a')\phi_b\phi_{b'}(a''), bb'b'') = (a\phi_b(a')\phi_{bb'}(a''), bb'b'')$$

where the second to last equality holds since $\phi_b : A \to A$ is a homomorphism and where the last equality holds since $\phi : B \to Aut(A)$ is a homomorphism. Thus $((a, b)(a', b'))(a'', b'') = (a, b)((a', b')(a'', b''))$, so that multiplication in $A \rtimes_\phi B$ is associative.

The definition of $\cdot$ in $A \rtimes_\phi B$ implies that for any $a \in A, b \in B$ in $A \rtimes_\phi B$ we have:

$$(a, b)(1, 1) = (1, 1)(a, b) = (a, b).$$

Thus $(1, 1)$ is the unit element in $A \rtimes_\phi B$. Using the fact that $\phi_b^{-1} = \phi_{b^{-1}}$ we see that

$$(a, b)(\phi_{b^{-1}}(a^{-1}), b^{-1}) = (a\phi_b\phi_{b^{-1}}(a^{-1}), bb^{-1}) = (aa^{-1}, bb^{-1}) = (1, 1),$$

and

$$(\phi_{b^{-1}}(a^{-1}), b^{-1})(a, b) = (\phi_{b^{-1}}(a^{-1})\phi_{b^{-1}}(a), b^{-1}b) = (\phi_{b^{-1}}(a^{-1}a), 1) = (1, 1).$$

Thus $(a, b)^{-1} = (\phi_{b^{-1}}(a^{-1}), b^{-1})$ in $A \rtimes_\phi B$, which completes verifying that $A \rtimes_\phi B$ is a group.

(2) Since A is normal in G, for every $b \in B$ we have $bAb^{-1} = A$. For $b \in B, a \in A$ denote $\phi_b(a) := bab^{-1} \in A$. It is easy to see that $\phi_b : A \to A$ is an isomorphism, so that $\phi_b \in Aut(A)$.

Define $f : A \rtimes_\phi B \to G$ as $f(a, b) := ab$, where $a \in A, b \in B$. We claim that f is an isomorphism.

Let $a, a' \in A, b, b' \in B$. Then in $A \rtimes_\phi B$ we have $(a, b)(a', b') = (a\phi_b(a'), bb')$ and

$$f(a, b)f(a', b') = aba'b' = a(ba'b^{-1})bb' = a\phi_b(a')bb' = f(a\phi_b(a'), bb').$$

Thus f is a homomorphism.

Suppose $f(a, b) = 1$. Then $f(a, b) = ab = 1$ and hence $a = b^{-1} \in A \cap B = \{1\}$. Therefore $a = 1, b = 1$. Thus $ker(f)$ is trivial and therefore f is injective. The assumption $G = AB$, together with the definition of f, imply that f is “onto”. Thus f is an isomorphism, as required.

Problem 4. Let $G \leq M = Isom(\mathbb{R}^2)$ be an abelian subgroup such that all elements of G are orientation-preserving isometries of $\mathbb{R}^2$.

Prove that either G consists of translations, or that there is some point $p \in \mathbb{R}^2$ such that G consists of rotations around p.

Solution.

By the classification theorem every orientation-preserving isometry of $\mathbb{R}^2$ is either a rotation or a translation.

If G consists only of translations, there is nothing to prove.

Suppose now that G contains a nontrivial rotation g around some point $p \in \mathbb{R}^2$.

Let $h \in G$ be an arbitrary nontrivial element. We know that h is either a nontrivial rotation or a nontrivial translation. Thus $gh(p) = h(p)$, so that g fixes $h(p)$. But g is a nontrivial rotation around p and the only point fixed by g is p. Hence $h(p) = p$. Therefore h cannot be a nontrivial translation (since then h would not fix anything) and hence h is a nontrivial rotation around some point q. Then the only point fixed by h is q and hence $p = q$, so that h is a rotation around p as required.

Problem 5.

Prove that for every $n \geq 3$ the center of S_n is $\{1\}$.

Solution.

Suppose that $\gamma \in S_n$ is a nontrivial element which belongs to the center of S_n. Then there are some $i \neq j$, $i, j \in \{1, \dots, n\}$ such that

$\gamma(i) = j$. Since $n \geq 3$, there is some number $k \in \{1, \ldots, n\}$ such that $k \neq i, k \neq j$.

Put $\alpha = (j\ k)$ to be the transposition exchanging j and k and fixing all the other elements of $\{1, \ldots, n\}$. In particular, $\alpha(i) = i$.

Put $\gamma' = \alpha\gamma\alpha^{-1}$.

Then

$$\gamma'(i) = \alpha\gamma\alpha^{-1}(i) = \alpha\gamma(i) = \alpha(j) = k.$$

On the other hand, by assumption γ belongs to the center of S_n and hence $\gamma' = \alpha\gamma\alpha^{-1} = \gamma$. However, $\gamma(i) = j \neq k$, yielding a contradiction.