Homework 11

**Problem 1** Let $R$ be a countable ring. Prove that the following conditions are equivalent:

(1) The ring $R$ is noetherian.
(2) There does not exist a strictly ascending infinite chain of ideals in $R$:

$$I_1 < I_2 < \cdots < I_n < \dots$$

**Problem 2** Let $R = \mathbb{Z}[x_1, x_2, \dots]$. Let $I = (x_1, x_2, \dots) \triangleleft R$. Prove that $I$ is not finitely generated.

**Problem 3** Let $S$ be a nonempty subset of $\mathbb{C}^n$.

Prove that there exist $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ such that $f_i|_S = 0$ for $1 \le i \le k$ and such that for every $g \in \mathbb{C}[x_1, \dots, x_n]$ satisfying $g|_S = 0$ there exist $g_1, \dots g_k \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$g = f_1 g_1 + \cdots + f_k g_k.$$

**Problem 4** Prove that if $R$ is a noetherian ring and $I \triangleleft R$ is an ideal in $R$ then the ring $R/I$ is also noetherian.

**Problem 5** Prove that if $R, S$ are noetherian ring then the ring $R \times S$ is also noetherian.

**Problem 6** Let $F$ be a field. An $F$-*algebra* is a vector space $R$ over $F$ such that $R$ is also a ring and such that

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \text{ for all } \alpha \in F, a, b \in R.$$

Let $R$ be an $F$-algebra which is finite-dimensional as a vector-space over $F$. Prove that $R$ is a noetherian ring.

**Problem 7**$^{**}$**[optional]** Prove that if $R$ is a noetherian ring then the ring of formal power series $R[[x]]$ is also noetherian.