**Problem 7**[optional]** Prove that if $R$ is a noetherian ring then the ring of formal power series $R[[x]]$ is also noetherian.

**Solution.**

For $f(x) = \sum_{i=0}^{\infty} c_i x^i$ define the *order* of $f$ as

$$or(f) := \min\{i | c_i \neq 0\} \text{ if } f \neq 0$$

and $or(f) := 0$ if $f = 0$. If $n = or(f)$ we call $c_n$ the *anti-leading coefficient* of $f$ and denote $AL(f) := c_n$.

Let $I \triangleleft R[[x]]$ be an arbitrary ideal. We need to show that $I$ is finitely generated. This is obvious for $I = 0$ so we will assume that $I \neq 0$.

Define

$$A = \{AL(f) : f \in I\} \subseteq R.$$

Note that $AL(0) = 0$ and hence $0 \in A$. The same argument as in the proof of Hilbert's Basis Theorem shows that:

**Lemma 1** We have $A \triangleleft R$.

Since $R$ is noetherian, the ideal $A$ is finitely generated and hence there exist some nonzero elements $a_1, \ldots, a_k \in R$ such that $A = (a_1, \ldots, a_k) \triangleleft R$. For each $j = 1, \ldots, k$ choose $f_j \in I$ such that $AL(f_j) = a_j$. By multiplying $f_j$ by powers of $x$ if necessary, we may assume that there is $n \geq 1$ such that $or(f_j) = n$ for $j = 1, \ldots, k$.

Consider the map $\pi_n : R[[x]] \to R[x]$ defined as $\pi_n(\sum_{i=0}^{\infty} c_i x^i) = \sum_{i=0}^{n-1} c_i x^i$. It is easy to see that $\pi_n$ is a homomorphism of $R$-modules and hence $\pi_n(I)$ is a submodule of the finitely generated $R$-module

$$P_n = \{f \in R[x] : deg(f) < n\}.$$

Since $P_n$ is a finitely generated $R$-module and $R$ is a noetherian ring, it follows that $\pi_n(I)$ is also a finitely generated $R$-module. Choose $h'_1, \ldots, h'_m \in P_n$ such that $\pi_n(I)$ is generated by $h'_1, \ldots, h'_m$ as an $R$-module. For each $j = 1, \ldots, m$ choose $h_j \in I$ such that $\pi_n(h_j) = h'_j$.

Thus for every $f \in I$ there exist $r_1, \ldots, r_m \in R$ such that

$$\pi_n(f) = \pi_n(r_1 h_1 + \cdots + r_m h_m),$$

That is, the power series $f$ and $r_1 h_1 + \ldots r_m h_m$ have the same coefficients for all powers of $x$ up to $x^{n-1}$.

We claim that

$$I = (h_1, \ldots, h_m, f_1, \ldots, f_k) \triangleleft R[[x]],$$

and hence $I$ is finitely generated, as required.

Let $g \in I$ be an arbitrary element. By the choice of $h_j$ there exist $r_1, \ldots, r_m \in R$ such that

$$\pi_n(g - r_1 h_1 - \cdots - r_m h_m) = 0.$$

We denote $f := g - r_1 h_1 - \cdots - r_m h_m$. Thus $or(f) \geq n$ and $f = \sum_{i=n}^{\infty} c_i x^i$ is an element of $I$.

contradict the assumption that $I$ is not finitely generated and therefore finish the proof):

To show that $I = (h_1, \ldots, h_m, f_1, \ldots, f_k)$ it suffices to prove:

**Lemma 2** We have

$$f \in (f_1, \ldots, f_k) \triangleleft R[[x]].$$

*Proof of Lemma 2.* We need to show that there exist elements $g_1, \ldots, g_k \in R[[x]]$ such that

$$f = f_1 g_1 + f_2 g_2 + \ldots f_k g_k \tag{*}$$

in $R[[x]]$.

Recall that $f = \sum_{i=n}^{\infty} c_i x^i$. Also, $or(f_j) = n$ and $f_j = \sum_{i=n}^{\infty} d_{i,j} x^i$.

Denote the coefficients of $g_j$ (which are yet to be proven to exist) by $g_j = \sum_{i=0}^{\infty} b_{i,j} x^i$.

By fully distributing the right-hand side of (*) we conclude that proving the existence of $g_1, \ldots, g_k \in R[[x]]$ satisfying (*) is equivalent to proving that there exist $\{b_{i,j} \in R : 1 \leq j \leq k, 0 \leq i < \infty\}$ such that:

$$\begin{aligned}
c_n &= \sum_{j=1}^{k} b_{0,j} d_{n,j} \\
c_{n+1} &= \sum_{j=1}^{k} b_{0,j} d_{n+1,j} + \sum_{j=1}^{k} b_{1,j} d_{n,j} \\
c_{n+2} &= \sum_{j=1}^{k} b_{0,j} d_{n+2,j} + \sum_{j=1}^{k} b_{1,j} d_{n+1,j} + \sum_{j=1}^{k} b_{2,j} d_{n,j} \\
&\vdots \\
c_{n+t} &= \sum_{p=0}^{t} \sum_{j=1}^{k} b_{p,j} d_{n+t-p,j} \\
&\vdots
\end{aligned} \tag{!}$$

for $t = 0, 1, 2, \ldots$.

We will prove by induction on $t = 0, 1, \ldots$ that for each $t \geq 0$ there exist

$$\{b_{p,j} \in R : 1 \leq j \leq k, 0 \leq p \leq t\}$$

such that the first $t + 1$ equations (up to $c_{n+t}$) in (!) hold.

**Base of Induction.** For $t = 0$ note that $c_n \in A$ since $f \in I$. Since $A = (a_1, \ldots, a_k) \triangleleft R$ and $f_j \in I$ has order $n$ and the anti-leading coefficient $a_j = d_{n,j}$, it follows that $c_n \in A = (d_{n,1}, \ldots, d_{n,k}) \triangleleft R$. Therefore there exist $b_{0,1}, \ldots, b_{k,1} \in R$ such that

$$c_n = b_{0,1} d_{n,1} + b_{0,2} d_{n,2} + \cdots + b_{0,k} d_{n,k} = \sum_{j=1}^{k} b_{0,j} d_{n,j}.$$

Thus we have found $b_{0,1}, \ldots, b_{k,1} \in R$ such that the first equation in (!) holds.

**Inductive Step.** Suppose now that $t > 0$ and we have constructed $\{b_{p,j} \in R : 1 \leq j \leq k, 0 \leq p \leq t - 1\}$ such that the first $t$ equations of (!) hold. We need to find $b_{t,1}, \ldots, b_{t,j}$ such that, together with the previously constructed $b_{i,j}$, the $t + 1$ equation of (!), involving $c_{n+t}$, holds.

Consider the power series

$$f' = f - \sum_{j=1}^{k} b_{0,j} f_j - \sum_{j=1}^{k} b_{1,j} x f_j - \cdots - \sum_{j=1}^{k} b_{t-1,j} x^{t-1} f_j.$$

By fully distributing the right-hand side in the above equality and computing the coefficients for all $x^i$, $i = 0, \ldots, n+t$ we observe the following. The fact that the equations of (!) involving $c_n, c_{n+1}, \ldots, c_{n+t-1}$ hold implies that $or(f') \geq n+t$. Moreover, the coefficient at $x^{n+t}$ in $f'$ is

$$c := c_{n+t} - \sum_{p=0}^{t-1} \sum_{j=1}^{k} b_{p,j} d_{n+t-p,j}.$$

By construction $f' \in I$ and therefore $c \in A = (d_{n,1}, \ldots, d_{n,k}) \triangleleft R$. Therefore the exist $b_{t,1}, \ldots, b_{t,j} \in R$ such that $c = \sum_{j=1}^{k} b_{t,j} d_{n,j}$ and hence

$$c_{n+t} = \sum_{p=0}^{t-1} \sum_{j=1}^{k} b_{p,j} d_{n+t-p,j} + \sum_{j=1}^{k} b_{t,j} d_{n,j} = \sum_{p=0}^{t} \sum_{j=1}^{k} b_{p,j} d_{n+t-p,j},$$

as required.

This concludes the proof of Lemma 2 and the proof that $R[[x]]$ is noetherian. $\square$