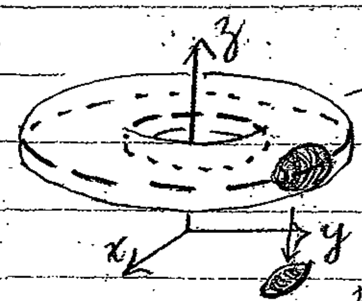
MA. 401, HOUR TEST #1 NAME: answers

1/ 

For which points p on this torus do (x,y) NOT give coordinates around p? Mark the points p.

projection to xy-plane is NOT 1-1 around points p on the outer & inner circles

2/ A 2-dimensional manifold M has a coordinate patch $\mathcal{U}, (u^1,u^2), \left\{\begin{matrix} u^1>0 \\ -\pi<u^2<\pi \end{matrix}\right\}$ and a map $F: \mathcal{U} \to \mathbb{R}^3$ satisfying

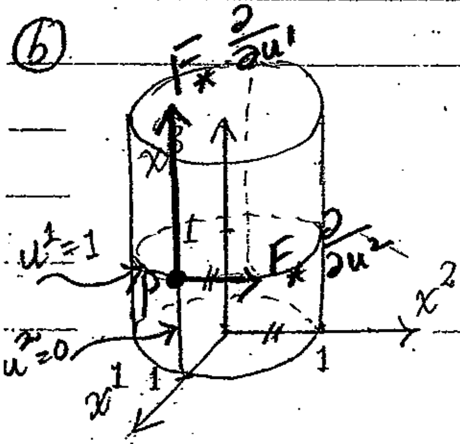
$$x^1 = \cos(u^2), \quad x^2 = \sin(u^2), \quad x^3 = (u^1)^2.$$

@ Express the $F_* \frac{\partial}{\partial u^i}$ in terms of the $\frac{\partial}{\partial x^j}$ at the point $p: u^1=1, u^2=0$

$$F_* \frac{\partial}{\partial u^1} = \frac{\partial x^i}{\partial u^1}\frac{\partial}{\partial x^i} = 2u^1 \frac{\partial}{\partial x^3}. \quad \text{At } u^1=1, u^2=0: \ 2\frac{\partial}{\partial x^3}$$

$$F_* \frac{\partial}{\partial u^2} = \frac{\partial x^i}{\partial u^2}\frac{\partial}{\partial x^i} = -\sin u^2 \frac{\partial}{\partial x^1} + \cos u^2 \frac{\partial}{\partial x^2}$$

$$\text{At } u^1=1, u^2=0: \ \frac{\partial}{\partial x^2}$$

(b) 

Sketch M in $\mathbb{R}^3$, marking the coordinate curves $u^1=1$ & $u^2=0$ through $p: (u^1,u^2)=(1,0)$, and the tangent vectors $F_* \frac{\partial}{\partial u^i}$ at p.

$p = \cancel{(\cos 0, \sin 0, 1)} = \cancel{(1,0,1)}$

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3/ NAME: ____________

A rod  moves freely in the plane. What is the configuration space? What is the dimension of this space?

$\mathbb{R}^2 \times S^1$

dimension 3

4/
$$M = \left\{ (x,y,z,u,v) \in \mathbb{R}^5 : \begin{array}{l} x^2+y^2+z^2+u^2+v^2 = 1 \\ 4x^2+4y^2+4z^2+4u^2+\frac{v^2}{4} = 1 \end{array} \right\}$$

ⓐ Use the IFT to show M is a manifold.

$F: \mathbb{R}^5 \to \mathbb{R}^2$: $F(x,y,z,u,v) = (x^2+y^2+z^2+u^2+v^2-1,\ 4x^2+4y^2+4z^2+4u^2+\frac{v^2}{4}-1)$.

$$DF = \begin{pmatrix} 2x & 2y & 2z & 2u & 2v \\ 8x & 8y & 8z & 8u & v/2 \end{pmatrix}.$$

The 2 equations defining M show that $v \neq 0$ at points in M.
Similarly, not all of x, y, z, u can be $=0$.
If $x \neq 0$, $\det\begin{pmatrix} 2x & 2v \\ 8x & v/2 \end{pmatrix} = xv - 16xv \neq 0$; similarly if $y \neq 0$ or $z \neq 0$, or $u \neq 0$

$\therefore$ rank $DF = 2$.

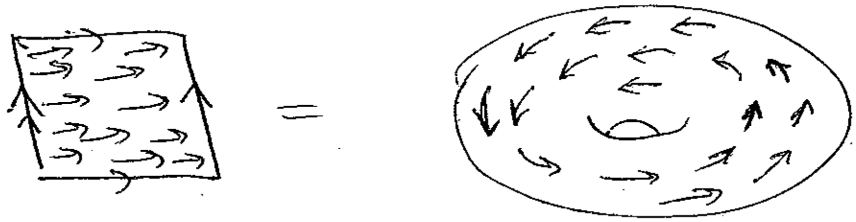
Note You must use the definition of M to show the rows are linearly independent — for other values of (x,y,z,u,v), they can be dependent!

ⓑ Give a choice from (x,y,z,u,v) that can be used as coordinates around $p = \left(\frac{1}{\sqrt5}, 0, 0, 0, \frac{2}{\sqrt5}\right) \in M$. Explain.

$$(DF)(p) = \begin{pmatrix} 2/\sqrt5 & 0 & 0 & 0 & 4/\sqrt5 \\ 8/\sqrt5 & 0 & 0 & 0 & 1/\sqrt5 \end{pmatrix}, \quad \det\begin{pmatrix} 2/\sqrt5 & 4/\sqrt5 \\ 8/\sqrt5 & 1/\sqrt5 \end{pmatrix} \neq 0$$

$\therefore (y, z, u)$ can be used as coordinates on M around p

5] a) Convince me by a picture that on a 2-torus $T^2 = S^1 \times S^1$ there does exist a nonvanishing continuous vector field

$T^2 =$ 

b) Use a) to explain why T^2 and S^2 are not diffeomorphic

The tangent map of a diffeomorphism takes a continuous nonvanishing vector field to a continuous nonvanishing vector field. Since T^2 has such a field but S^2 does not, they are not diffeomorphic

c) Let M be a 2-manifold. If $(U, \varphi = (x,y))$ and $(U', \psi = (z,w))$ are coordinate patches ~~on~~ M, write $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial w} = \frac{\partial x}{\partial w}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y}\frac{\partial}{\partial y}$$

6] Let $\mathbb{P}^2$ be the projective plane, that is, the set of all lines through the origin in $\mathbb{R}^3$.

Consider the function

$$f : \mathbb{P}^2 \longrightarrow \mathbb{R}$$

$$f([x:y:z]) = \frac{2x}{x+y+z}$$

a) Consider the chart (U, φ) where $U = \{ [x:y:z] \in \mathbb{P}^2 \mid z \neq 0 \}$

$$\varphi([x;y:z]) = (\underbrace{\tfrac{x}{z}}_{\shortparallel \atop u^1}, \underbrace{\tfrac{y}{z}}_{\shortparallel \atop u^2})$$

Compute df explicitly in this chart

$$\varphi^{-1}(u^1,u^2) = [u^1 : u^2 : 1]$$

$$(f \circ \varphi^{-1})(u^1,u^2) = \frac{2u^1}{u^1+u^2+1}$$

$$df = \frac{\partial f}{\partial u^1} du^1 + \frac{\partial f}{\partial u^2} du^2 =$$

$$= \frac{2u^2+2}{(u^1+u^2+1)^2} du^1 - \frac{2u^1}{(u^1+u^2+1)^2} du^2$$

b) consider the map $\gamma : \mathbb{R}_{>0} \longrightarrow \mathbb{P}^2$

given by $\gamma(t) = [1 : t^2 : t]$, $t > 0$

compute $\gamma_*(\frac{\partial}{\partial t})$ explicitly in the above chart.

$$(\varphi \circ \gamma)(t) = \left(\frac{1}{t}, \frac{t^2}{t} \right) = \Big(\underset{\substack{\shortparallel \\ u^1}}{\frac{1}{t}}, \underset{\substack{\shortparallel \\ u^2}}{t} \Big)$$

$$\gamma_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial u^1}{\partial t} \frac{\partial}{\partial u^1} + \frac{\partial u^2}{\partial t} \frac{\partial}{\partial u^2} =$$

$$= -\frac{1}{t^2} \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2}$$