Math 481

13. Riemannian Metrics

Reference: Frankel, pp 42-47

1. a) A *Riemannian metric* $g = \langle , \rangle$ in M is a positive definite, symmetric twice covariant tensor field, i.e. for $\vec{v}, \vec{w} \in M_p$,

   - $g(\vec{v}, \vec{w}) = g(\vec{w}, \vec{v})$. If there is only one g under consideration, we often write this as $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.
   - $g(\vec{v}, \vec{v}) \geq 0$, and is 0 if and only if $\vec{v} = 0$.

   b) In a coordinate patch, g has components

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ji}.$$

   c) Recall the transformation law of covariant tensors on patch overlaps:

$$g'_{ij} = \frac{\partial x^k}{\partial x'^l}\frac{\partial x^l}{\partial x'^j} g_{kl}.$$

2. a) Note: A Riemannian metric gives a coordinate independent way to identify M_p and M_p^*; namely for $\vec{v} \in M_p$, define $\alpha_{\vec{v}} \in M_p^*$ by $\alpha_{\vec{v}}(\vec{w}) = \langle \vec{v}, \vec{w} \rangle$.

   b) In coordinates, $\vec{v} = v^i \frac{\partial}{\partial x^i}$ and

$$\alpha_{\vec{v}}\left(\frac{\partial}{\partial x^j}\right) = \left\langle v^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = v^i g_{ij}.$$

So we are mapping the tangent vector $\vec{v}$ with components $(v^1, \ldots, v^n)$ to the 1-form $\alpha_{\vec{v}} = v_j dx^j$ with components $(v^i g_{i1} = v_1, \ldots, v^i g_{in} = v_n)$.

3. a) We know that $\alpha_{\vec{v}}$ is a 1-form, so its components $v_1, \ldots, v_n$ transform on patch overlaps by the law: $v'_i = \frac{\partial x^j}{\partial x'^i} v_j$. To check this directly,

$$\begin{aligned} v'_i &= v'^j g'_{ij} = \left(\frac{\partial x'^j}{\partial x^k} v^k \right) \left(\frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} g_{lm} \right) \\ &= \left(\frac{\partial x'^j}{\partial x^k} \frac{\partial x^m}{\partial x'^j} \right) \frac{\partial x^l}{\partial x'^i} v^k g_{lm} = \delta^m_k \frac{\partial x^l}{\partial x'^i} v^k g_{lm} \\ &= \frac{\partial x^l}{\partial x'^i} v^k g_{lk} = \frac{\partial x^l}{\partial x'^i} v_l \end{aligned}$$

b) On the first line, we used:

If $\vec{v} = v^i \frac{\partial}{\partial x^i} \in M_p$, and we change coordinates so $\vec{v} = v'^j \frac{\partial}{\partial x'^j}$, then $v'^j = \frac{\partial x'^j}{\partial x^k} v^k$.

Reason: $\vec{v} = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j}$. Another way to say this: tangent vectors are 1-contravariant tensors.

4. a) Let g be a Riemannian metric on M. In a patch, we define g^{ij} by $(g^{ij})_{m \times m} = (g_{ij})^{-1}_{m \times m}$. That is, $g^{ij} g_{jk} = \delta^i_k$.

b) What transformation law do the g^{ij} satisfy under change of coordinates? We know

$$g'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}$$

and so

$$(g'_{ij}) = \left(\frac{\partial x^k}{\partial x'^i} \right)^T (g_{kl}) \left(\frac{\partial x^l}{\partial x'^j} \right).$$

Therefore,

$$\begin{aligned} (g'^{ij}) &= \left(\frac{\partial x^l}{\partial x'^j} \right)^{-1} (g_{kl})^{-1} \left(\left(\frac{\partial x^k}{\partial x'^i} \right)^T \right)^{-1} \\ &= \left(\frac{\partial x'^j}{\partial x^l} \right) (g^{kl}) \left(\frac{\partial x'^i}{\partial x^k} \right)^T \end{aligned}$$

i.e. $g'^{ij} = \frac{\partial x'^j}{\partial x^l} \frac{\partial x'^i}{\partial x^k} g^{kl}$. So, the g^{ij} are the components of a twice contravariant tensor.

5. a) For $f : M \to \mathbb{R}$, df is a 1-form, given in a chart by $df = \frac{\partial f}{\partial x^i} dx^i$. That is, $\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}$ transform as a 1-form, not as a tangent vector. In other words, df is the correct analogue on manifolds of "grad f".

b) If you really want a vector field "grad f", you can get it by converting df to a contravariant tensor, using a Riemannian metric g. Without g, there is no chart independent way to get "grad f". Using g, you can define grad$f$ to have components

$$v^i = g^{ij} \frac{\partial f}{\partial x^j}.$$