Math 481

17. $\int_\gamma \alpha$.

- We want to define $\int_\gamma \alpha$, for any r-form α and singular r-cube γ in a manifold M:

a) Definition: In coordinates $(U, (x^1, \ldots, x^n))$ on M, we have $\gamma : [0,1]^r \to M$, $\gamma(u^1, \ldots, u^r) = (x^1(u^1, \ldots, u^r), \ldots, x^n(u^1, \ldots, u^r))$, and $\alpha = \sum_{\vec{I}=(1\le i_1<\ldots<i_r\le n)} a_{\vec{I}}\, dx^{\vec{I}}$. So

$$\begin{aligned}
\gamma^*\alpha &= \sum_{\vec{I}} a_{\vec{I}} \left(\sum_{j_1=1}^{r} \frac{\partial x^{i_1}}{\partial u^{j_1}} du^{j_1} \right) \wedge \ldots \wedge \left(\sum_{j_r=1}^{r} \frac{\partial x^{i_r}}{\partial u^{j_r}} du^{j_r} \right) \\
&= \sum_{\vec{I}} a_{\vec{I}} \det \frac{\partial(x^{i_1}, \ldots, x^{i_r})}{\partial(u^1, \ldots, u^r)} du^1 \wedge \ldots \wedge du^r
\end{aligned}$$

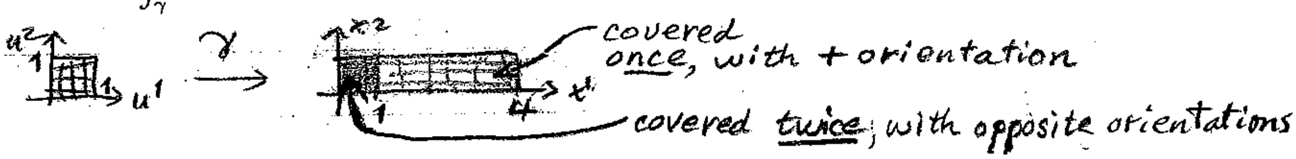
So, now that we've written $\gamma^*\alpha = f(u^1, \ldots, u^r) du^1 \wedge \ldots \wedge du^r$, just define

$$\int_\gamma \alpha = \int_0^1 \cdots \int_0^1 f(u^1, \ldots, u^r)\, du^1 \ldots du^r.$$

b) Example: Line integrals, see Handout #15 d).

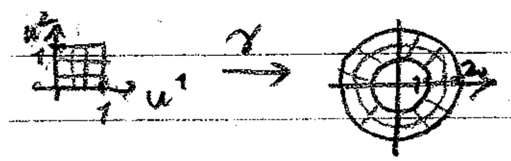
c) Example: $M = \mathbb{R}^2$. Take the 2-form $\alpha = dx^1 \wedge dx^2$ (the Euclidean "area form") and the singular 2-cube $\gamma(u^1, u^2) = (9(u^1 - \frac{1}{3})^2, u^2)$, $0 \le u^i \le 1$. Then

$\int_\gamma \alpha = 3.$



d) Example: $M = \mathbb{R}^2$, $\alpha = dx^1 \wedge dx^2$, $\gamma(u^1, u^2) = ((u^1+1)\cos(2\pi u^2), (u^1+1)\sin(2\pi u^2))$.

$\int_\gamma \alpha$ should be $\pi(2)^2 - \pi(1)^2 = 3\pi$. Check this!



e) In d), $\alpha = d(\frac{1}{2}(x^1 dx^2 - x^2 dx^1)) = d\beta$, where $\beta = \frac{1}{2}(x^1 dx^2 - x^2 dx^1)$. So,

$$\int_\gamma \alpha = \int_\gamma d\beta = 3\pi.$$

Now, calculate $\int_{\partial\gamma} \beta$. We have

$$\partial\gamma = \gamma_{2,0} + \gamma_{1,1} - \gamma_{2,1} - \gamma_{1,0} = \gamma_{1,1} - \gamma_{1,0},$$

where

$$\begin{aligned}
\gamma_{2,0}(u^1) = (u^1+1, 0),\ \gamma_{1,1}(u^2) = (2\cos(2\pi u^2), 2\sin(2\pi u^2)), \\
\gamma_{2,1}(u^1) = (u^1+1, 0),\ \gamma_{1,0}(u^2) = (\cos(2\pi u^2), \sin(2\pi u^2)).
\end{aligned}$$

Check that the answer is ... $3\,\pi$!

4.6. Maxwell's Field Equations

In classical electromagnetic field theory one deals with the following quantities:

$\mathbf{E}$ = electric field	$\mathbf{H}$ = magnetic field
$\mathbf{B}$ = magnetic induction	$\mathbf{J}$ = electric current density
$\mathbf{D}$ = dielectric displacement	ρ = charge density.

These are all functions of the space variables x^1, x^2, x^3 and the time t. The basic Maxwell equations in ordinary vector language are

(i) $\operatorname{curl} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t}$ (Faraday's law of induction)

(ii) $\operatorname{curl} \mathbf{H} = \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\dfrac{\partial \mathbf{D}}{\partial t}$ (Ampère's law)

(iii) $\operatorname{div} \mathbf{D} = 4\pi\rho$ (continuity)

(iv) $\operatorname{div} \mathbf{B} = 0$ (nonexistence of true magnetism)

Here c is the speed of light. We shall put these equations into the language of exterior forms. To this end, we set

forms on $\mathbb{R}^4 = \{(x^1, x^2, x^3, t)\}$

$$\begin{aligned}
\alpha &= (E_1\,dx^1 + E_2\,dx^2 + E_3\,dx^3)(c\,dt) \\
&\qquad + (B_1\,dx^2\,dx^3 + B_2\,dx^3\,dx^1 + B_3\,dx^1\,dx^2), \\
\beta &= -(H_1\,dx^1 + H_2\,dx^2 + H_3\,dx^3)(c\,dt) \\
&\qquad + (D_1\,dx^2\,dx^3 + D_2\,dx^3\,dx^1 + D_3\,dx^1\,dx^2), \\
\gamma &= (J_1\,dx^2\,dx^3 + J_2\,dx^3\,dx^1 + J_3\,dx^1\,dx^2)\,dt - \rho\,dx^1\,dx^2\,dx^3.
\end{aligned}$$

c = constant
($c = 1$ ok)

Equations (i) and (iv) become

$$d\alpha = 0.$$

Equations (ii) and (iii) become

$$d\beta + 4\pi\gamma = 0.$$

Applying d to this last equation yields

$$d\gamma = 0,$$

in vector notation

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

From the equation $d\alpha = 0$ one concludes, at least in any region of space-time which can be shrunken to a point, that there is a one-form λ such that

$$d\lambda = \alpha.$$

We introduce the *vector* potential $\mathbf{A}$ and a scalar A_0 by writing

$$\lambda = A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3 + A_0 c\,dt.$$

The equation $d\lambda = \alpha$ in vector form is

$$\begin{cases} \operatorname{curl} \mathbf{A} = \mathbf{B} \\ \operatorname{grad} A_0 - \dfrac{1}{c}\dfrac{\partial \mathbf{A}}{\partial t} = \mathbf{E}. \end{cases}$$

from Flanders, Differential Forms with Applications to the Physical Sciences

In *free space*, everything simplifies according to

$$\mathbf{E} = \mathbf{D}, \qquad \mathbf{H} = \mathbf{B},$$
$$\mathbf{J} = 0, \qquad \rho = 0$$

so that the Maxwell equations become

$$\begin{cases} \operatorname{curl} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} & \operatorname{div} \mathbf{E} = 0 \\ \operatorname{curl} \mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} & \operatorname{div} \mathbf{H} = 0. \end{cases}$$

We introduce the *Lorentz metric* into 4-space whereby

$$dx^1, dx^2, dx^3, c\,dt$$

is an orthonormal basis:

$$(dx^i, dx^j) = \delta^{ij}, \qquad (dx^i, c\,dt) = 0,$$
$$(c\,dt, c\,dt) = -1.$$

The signature is $3 - 1 = 2$.

According to the formulas of Section 2.7,

$$*(dx^1\,dx^2) = -dx^3(c\,dt), \qquad \text{etc.},$$
$$*(dx^1\,c\,dt) = dx^2\,dx^3, \qquad \text{etc.}$$

We see that

$$\begin{aligned} \alpha &= (E_1\,dx^1 + \cdots)(c\,dt) + (H_1\,dx^2\,dx^3 + \cdots), \\ \beta &= -(H_1\,dx^1 + \cdots)(c\,dt) + (E_1\,dx^2\,dx^3 + \cdots) \\ &= *\alpha. \end{aligned}$$

Consequently Maxwell's equations in free space are simply

$$\begin{cases} d\alpha = 0 \\ d*\alpha = 0. \end{cases}$$

We return to the general situation and refine our analysis by introducing one-forms:

$$\begin{aligned} \omega_1 &= E_1\,dx^1 + E_2\,dx^2 + E_3\,dx^3 \\ \omega_2 &= B_1\,dx^2\,dx^3 + B_2\,dx^3\,dx^1 + B_3\,dx^1\,dx^2 \\ \omega_3 &= H_1\,dx^1 + H_2\,dx^2 + H_3\,dx^3 \\ \omega_4 &= D_1\,dx^2\,dx^3 + D_2\,dx^3\,dx^1 + D_3\,dx^1\,dx^2 \\ \omega_5 &= J_1\,dx^2\,dx^3 + J_2\,dx^3\,dx^1 + J_3\,dx^1\,dx^2. \end{aligned}$$

These involve space variable differentials only. Now we interpret d' to denote the *exterior derivative with respect to space variables only.* We introduce $\partial/\partial t$ in this form

$$\frac{\partial}{\partial t}(\omega_1) = \dot{\omega}_1 = \dot{E}_1\,dx^1 + \cdots, \text{etc.}$$

Now the Maxwell equations are

$$\begin{cases} d'\omega_1 = -\dfrac{1}{c}\,\dot{\omega}_2 \\ d'\omega_3 = \dfrac{4\pi}{c}\,\omega_5 + \dfrac{1}{c}\,\dot{\omega}_4 \\ d'\omega_2 = 0 \\ d'\omega_4 = 4\pi\rho\,dx^1\,dx^2\,dx^3. \end{cases}$$

The *Poynting energy-flux vector* $\mathbf{S}$ is introduced by

$$\mathbf{S} = \left(\frac{c}{4\pi}\right)\mathbf{E} \times \mathbf{H}$$

that is

$$\left(\frac{c}{4\pi}\right)\omega_1 \wedge \omega_3 = S_1\,dx^2\,dx^3 + S_2\,dx^3\,dx^1 + S_3\,dx^1\,dx^2.$$

Poynting's theorem,

$$\left(\frac{1}{4\pi}\right)\dot{\mathbf{B}}\cdot\mathbf{H} + \mathbf{E}\cdot\mathbf{J} + \left(\frac{1}{4\pi}\right)\mathbf{E}\cdot\dot{\mathbf{D}} + \operatorname{div}\mathbf{S} = 0,$$

follows from

$$\begin{aligned} d'(\omega_1 \wedge \omega_3) &= d'\omega_1 \wedge \omega_3 - \omega_1 \wedge d'\omega_3 \\ &= \left(-\frac{1}{c}\,\dot{\omega}_2\right) \wedge \omega_3 - \omega_1 \wedge \left(\frac{4\pi}{c}\,\omega_5 + \frac{1}{c}\,\dot{\omega}_4\right) \\ &= -\frac{1}{c}\,\dot{\omega}_2 \wedge \omega_3 - \frac{4\pi}{c}\,\omega_1 \wedge \omega_5 - \frac{1}{c}\,\omega_1 \wedge \dot{\omega}_4. \end{aligned}$$

For bodies at rest, one assumes $\mathbf{D} = \kappa\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$ where the *dielectric constant* κ and the *permeability* μ are constant in time. Then Poynting's theorem becomes

$$-\frac{\partial u}{\partial t} = \operatorname{div}\mathbf{S} + \mathbf{E}\cdot\mathbf{J}$$

where

$$u = \frac{1}{8\pi}(\kappa\mathbf{E}^2 + \mu\mathbf{H}^2)$$

is the *energy density* of the field. The quantity $\mathbf{E}\cdot\mathbf{J}$ is called the *therm* *chemical activity*.

4.7. Problems

1. Develop the formula for the Laplacian in cylindrical coordinates.

2. A complex matrix A is *unitary* if $AA^* = I$, where $A^* = {}^t\bar{A}$, tl transpose conjugate of A. We call A skew-hermitian if $A^* + A =$ Discuss the connection between unitary and skew-hermitian matrices.

3. Show that e^A is orthogonal if A is skew-symmetric. Here

$$e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

for the real matrix A.

4. Set up a frame and the structure equations for a sphere of radius Compute the curvatures.

5. Find Gaussian curvature of the surface of revolution obtained revolving the curve

$$\begin{cases} x = \cos\theta + \ln\tan(\theta/2) \\ y = \sin\theta \\ \dfrac{\pi}{2} < \theta < \pi \end{cases}$$

about the x-axis.

6. Given a surface in the form $z = f(x, y)$, develop formulas for H and in terms of f and its partial derivatives.

7. Let Σ be a surface. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_1'$, $\mathbf{e}_2'$ be two moving frames tangent vectors to Σ. Determine the relation between the correspondi ϖ and ϖ' and verify that $d\varpi = d\varpi'$.

8. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ be two moving frames in $\mathbf{E}^3$. Set the orthogonal matrix relating these frames and determine how the corr sponding Ω and Ω' are related.