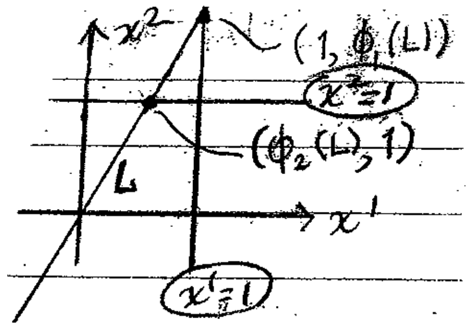
Math 481
3. Checking the manifold definition

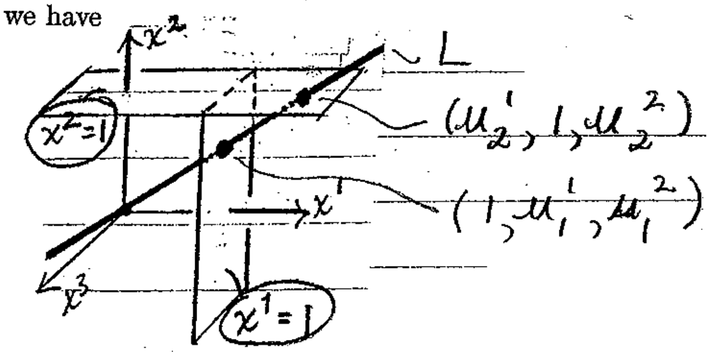
1. Example: An atlas on P^1. Recall that P^1 is the space of all lines in $\mathbb{R}^2$ through $(0,0)$.



The map ϕ_i is defined on what set $U_i \subset P^1$? Is (U_i, ϕ_i) a coordinate patch? Are $(U_1, \phi_1), (U_2, \phi_2)$ an atlas?

- Is every line through $(0,0)$ contained in some U_i?
- Is $\phi_1(U_1 \cap U_2)$ open in $\mathbb{R}^2$? $\phi_2(U_1 \cap U_2)$?
- What is $\phi_2 \circ \phi_1^{-1}$? $\phi_1 \circ \phi_2^{-1}$?

2. Similarly, on Worksheet 1, for P^2 we have
   - $\phi_1(L) = (u_1^1, u_1^2)$,
   - $\phi_2(L) = (u_2^1, u_2^2)$,
   - $\phi_3(L) = ?$



3. Example: Consider k numbered rods of length 1, in a closed chain with hinged joints, in the plane. What is the configuration space if $k = 3$,

   a) assuming the first rod is fixed,

   b) no constraints assumed?

Math 481
4. Orientability

1. A manifold is *orientable* if it has an atlas such that whenever $U_i \cap U_j \neq \emptyset$, then $D(\phi_i \circ \phi_j^{-1})$ has positive determinant.

   a) Here, $\phi_i = (u_i^1, \ldots, u_i^n) : U_i \to \mathbb{R}^n$ and $D(\phi_i \circ \phi_j^{-1})$ is the Jacobian

$$\begin{pmatrix} \frac{\partial u_i^1}{\partial u_j^1} & \cdots & \frac{\partial u_i^1}{\partial u_j^n} \\ \vdots & & \vdots \\ \frac{\partial u_i^n}{\partial u_j^1} & \cdots & \frac{\partial u_i^n}{\partial u_j^n} \end{pmatrix}$$

   which we also write as

$$\frac{\partial(u_i^1, \ldots, u_i^n)}{\partial(u_j^1, \ldots, u_j^n)}$$

   b) on $U_i \cap U_j$, $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are inverse maps, so, by the Multivariable Chain Rule, their Jacobians are inverse matrices. Hence, their determinants are inverse numbers (remember for $n \times n$ matrices A, B, $\det(AB) = \det(A)\det(B)$). Thus, these determinants are nonzero.

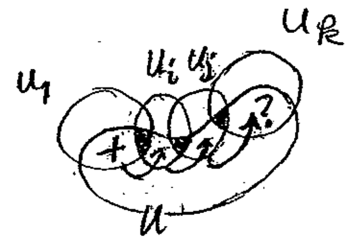
   c) Since $\det D(\phi_i \circ \phi_j^{-1}) \neq 0$, it has the same sign at all the points in a connected component of $U_i \cap U_j$.

   d) Note: switching any pair of ϕ_i coordinates switches two rows of $D(\phi_i \circ \phi_j^{-1})$. Switching any pair of ϕ_j coordinates switches two columns of $D(\phi_i \circ \phi_j^{-1})$. Either switch changes the sign of the determinant.

2. A <u>2-dimensional</u> manifold is orientable if and only if it does not contain a Möbius band. For example, P^2 is <u>not</u> orientable.

3. a) Suppose M is an orientable manifold and $\mathcal{A} = \{(U_\alpha, \phi_\alpha = (u_\alpha^1, \ldots, u_\alpha^n))\}$ is an *orienting atlas* ($\det \dfrac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} > 0$ on $U_\alpha \cap U_\beta$). For any other compatible coordinate patch $(U, \phi = (x^1, \ldots, x^n))$ with U connected, the sign of $\det \dfrac{\partial(x^1, \ldots, x^n)}{\partial(u_\alpha^1, \ldots, u_\alpha^n)}$ is the same for all U_α with $U \cap U_\alpha \neq \emptyset$.

Reason: by the Multivariable Chain Rule, we have



$$\frac{\partial(x^1, \ldots, x^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} = \frac{\partial(x^1, \ldots, x^n)}{\partial(u_\alpha^1, \ldots, u_\alpha^n)} \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)}$$

and det is multiplicative.

b) Conclusion: If a manifold M has a finite atlas of connected coordinate patches, there is a finite recursive procedure for deciding if M is orientable: Start with the first patch and try to alter all those that intersect it to get the determinant of the Jacobians to be positive. If this cannot be done, M is not orientable. If it can, repeat starting with the second coordinate patch.

<u>Reference</u>: Frankel, pp. 82-85.