Math 481
5. Implicit Function Theorem

1. Solutions of Constraint Equations

   a) Simplest Case: $M = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^{n+1} = h(x^1, \ldots, x^{n+1})\}$.

   ($M =$ graph of h.) M is a manifold, with an atlas consisting of one coordinate patch.

   b) Now let $M = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : g(x^1, \ldots, x^{n+1}) = 0\}$. Suppose $Dg \neq (0, \ldots, 0)$ at every $p \in M$. Then, the *Implicit Function Theorem* states that

   (i) M is an n-dimensional manifold.

   (ii) Around any $p \in M$, there is a coordinate patch $(\tilde{W}, \phi : \tilde{W} \to \mathbb{R}^n)$ where ϕ is projection onto an open set W in the $(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1})$ plane. Here, x^i is any coordinate such that $\frac{\partial g}{\partial x^i}(p) \neq 0$.

   (iii) On W, $x^i \circ \phi^{-1}$ is a smooth function $h(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1})$, so $\tilde{W}$ is the graph of $h : W \to \mathbb{R}$ (referred to as the "implicit function").

2. Example: $M =$ unit sphere in $\mathbb{R}^{n+1}$.

   Simple guiding example: $M = S^1 \subset \mathbb{R}^2$.

3. Implicit Function Theorem (General Form) :

Assume : a) Consider the solution set of k equations in $n+k$ variables:
For an open set U in $\mathbb{R}^{n+k}$, let

$$M = \left\{ (x^1, \dots, x^{n+k}) \in U : g^1(x^1, \dots, x^{n+k}) = 0, \dots, g^k(x^1, \dots, x^{n+k}) = 0 \right\}$$

i.e. $g(x^1, \dots, x^{n+k}) = \left(g^1(x^1, \dots, x^{n+k}), \dots, g^k(x^1, \dots, x^{n+k})\right)$ defines a map $g : U \to \mathbb{R}^k$ and $M = g^{-1}(0, \dots, 0)$.

b) Suppose that rank $Dg = k$ at every $y \in M$. This means that the rows of

$$Dg = \begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \cdots & \frac{\partial g^1}{\partial x^{n+k}} \\ \vdots & & \vdots \\ \frac{\partial g^k}{\partial x^1} & \cdots & \frac{\partial g^k}{\partial x^{n+k}} \end{pmatrix}$$

are linearly independent. This happens if and only if the only solution of the equation

$$c_1(\text{row}_1) + \dots + c_k(\text{row}_k) = 0$$

is $c_1 = \dots = c_k = 0$. Equivalently, for some choice of k columns of Dg, the corresponding $k \times k$ subdeterminant is nonzero. We say that " g is a submersion at every $y \in M$ ".

Conclusion : a) M is an n-dimensional manifold.

b) At any $y \in M$, you can get a chart around y as follows: Take any choice of k columns , say columns $j_1, \dots, j_k$, of $Dg(y)$, so that the $k \times k$ matrix they form has nonzero determinant.

Then there is a chart $(\tilde{W}, \phi)$ around $y \in M$, where $\phi : \tilde{W} \to \mathbb{R}^n$, is projection onto an open set W in the coordinate plane of the remaining n variables $(x^1, \dots, \widehat{x}^{j_1}, \dots, \widehat{x}^{j_k}, \dots, x^{n+k})$.

c) On W, the implicit functions $x^{j_1} \circ \phi^{-1}, \dots, x^{j_k} \circ \phi^{-1}$ are smooth functions $W \to \mathbb{R}$:

$$\begin{aligned} x^{j_1} \circ \phi^{-1} &= h^{j_1}(x^1, \dots, \widehat{x}^{j_1}, \dots, \widehat{x}^{j_k}, \dots, x^{n+k}) \\ &\vdots \\ x^{j_k} \circ \phi^{-1} &= h^{j_k}(x^1, \dots, \widehat{x}^{j_1}, \dots, \widehat{x}^{j_k}, \dots, x^{n+k}) \end{aligned}$$