Math 481
8. Diffeomorphisms and Homeomorphisms

1. a) Manifolds M and N are *diffeomorphic* if there exists a one to one, onto map $F : M \to N$ such that both F and F^{-1} are smooth in patch coordinates.

   b) That is, if (U, ϕ) is a patch at $p \in M$, and (V, ψ) is a patch at $F(p) \in N$, then $(\psi \circ F \circ \phi^{-1})$ and $(\phi \circ F^{-1} \circ \psi^{-1})$ are smooth.

   c) You only have to check this for one choice of an atlas on M and an atlas on N. Then it holds for all

   other compatible charts.

   d) Note: If we substitute "continuous" for "smooth" in our definitions of a manifold and diffeomorphism,

   we get the definitions of "topological manifold" and "homeomorphism". Thus, diffeomorphic manifolds

   are homeomorphic.

   e) An n-dimensional topological manifold, $n \leq 3$, can be given a unique differentiable structure.

   But if $n \geq 4$, there are topological manifolds that cannot be given a smooth structure.

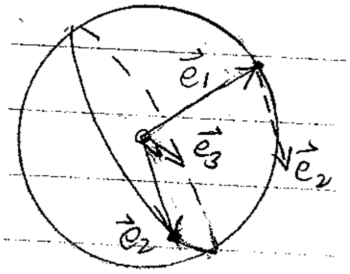
2. Example: $\mathbb{R}^3, S^3, S^2 \times S^1, S^1 \times S^1 \times S^1, P^3, SO(3)$: all of dimension 3.

   a) There are "topological invariants" that show every pair except the last are *not* homeomorphic.

   An "extra problem" will be to show that P^3 and $SO(3)$ *are* homeomorphic.

   b) At first you might think that $SO(3)$ and $S^2 \times S^1$ are homeomorphic: just choose $\vec{e}_1 \in S^2$. Then

   for each $\vec{e}_1$, choose $\vec{e}_2 \in S^1$. What is wrong with this argument?



Math 481
9. The Flow of a Vector Field

1. Flow Lines of a Vector Field [Frankel, pp. 30-35]

   a) If Y is a vector field on M, then an *integral curve* (or *flow line*) of Y is a curve $\gamma(t)$ with $\frac{d\gamma}{dt} = Y_{\gamma(t)}$.

   b) In a coordinate patch, if $Y = \sum_{i=1}^{n} Y^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}$, then $\gamma(t) = (x^1(t), \ldots, x^n(t))$ is obtained by solving the system of ODEs

$$\frac{dx^i}{dt} = Y^i(x^1(t), \ldots, x^n(t)),\ 1 \le i \le n.$$

   c) Example: $M = \mathbb{R}^2$, $Y = \frac{\partial}{\partial x^1} + (x^2)^2\frac{\partial}{\partial x^2}$.
   So, $Y^1(x^1, x^2) = 1$ and $Y^2(x^1, x^2) = (x^2)^2$. The flow lines $\gamma(t) = (x^1(t), x^2(t))$ are solutions of $\frac{dx^1}{dt} = 1$, $\frac{dx^2}{dt} = (x^2)^2$. Thus,

$$x^1 = t + a, x^2 = \frac{1}{\frac{1}{b} - t}$$

   is the flow line satisfying $\gamma(0) = (a, b), b \neq 0$ and

$$x^1 = t + a, x^2 = 0$$

   is the flow line satisfying $\gamma(0) = (a, 0)$.

   d) Theorem There is a unique maximal (i.e. defined on the largest possible t interval) integral curve γ_p with $\gamma_p(0) = p$.

2. Local Flows

   a) Let $\phi_t(p) = \gamma_p(t)$ (if the latter exists). This map moves every point of M along a flow line of Y for a time t.

   b) It may not be possible to flow every point for a given time $t \neq 0$ (see the above example). But, every point p lies in a chart neighborhood U that can be flowed for some (possibly small) fixed time; i.e. $\phi_t(q)$ exists for all $q \in U$ and $-\epsilon < t < \epsilon$ (this is the Fundamental Theorem on p. 32 of Frankel: we say "local flows exist").

3. Local Flows of γ at a nonsingular point (i.e. $Y_p \neq 0$.)

   a) Theorem: If $Y_p \neq 0$, there is a chart $(U, (y^1, \ldots, y^n))$ around p such that $\frac{\partial}{\partial y^1}|_p = Y_q$ for all $q \in U$.

Math 481
10. Tangent Maps (Pushforward)

1. a) A smooth map $F : M^m \to N^n$ determines a linear *tangent map* of tangent spaces

$$F_* : M_p \to N_{F(p)}.$$

b) For coordinates $\phi = (x^1, \ldots, x^m)$ at $p \in M$, and $\psi = (y^1, \ldots, y^n)$ at $F(p) \in N$,

$$F_* \left(\frac{\partial}{\partial x^j} \right) = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

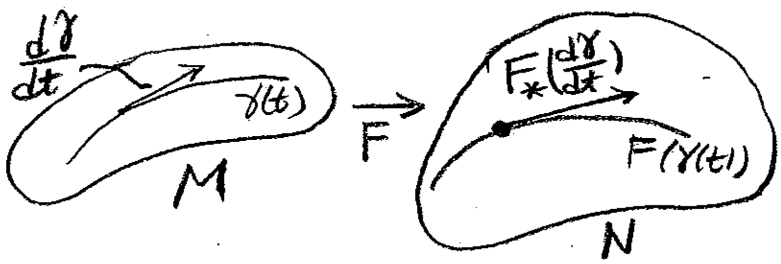
i.e. the matrix of the linear transformation F_* is the Jacobian

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^m)} = D(\psi \circ F \circ \phi^{-1}).$$

c) We can check directly that the definition of F_* does not depend on the choice of coordinates.

Or, we can find a coordinate independent definition. Let $\gamma(t)$ be a curve in M and $\frac{d\gamma}{dt}$ a tangent vector tangent to $\gamma(t)$. Define

$$F_* \left(\frac{d\gamma}{dt} \right) = \frac{d}{dt} F(\gamma(t)).$$



Then the formula above for $F_* \left(\frac{\partial}{\partial x^j} \right)$ follows directly from the chain rule. Explain:

2. Example: F : upper half plane $\to$ upper half plane, $F(x^1, x^2) = (y^1, y^2)$ where $y^1 = \frac{x^1}{x^2}$ and $y^2 = x^2$. We have

$$F_* \frac{\partial}{\partial x^1} = \frac{\partial y^1}{\partial x^1}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial}{\partial y^2} = \frac{1}{x^2}\frac{\partial}{\partial y^1} \tag{1}$$

$$F_* \frac{\partial}{\partial x^2} = \frac{\partial y^1}{\partial x^2}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^2}\frac{\partial}{\partial y^2} = \frac{-x^1}{(x^2)^2}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}. \tag{2}$$

Or we may view equations (1) and (2) above as

$$\begin{aligned}\left(F_* \tfrac{\partial}{\partial x^1} \quad F_* \tfrac{\partial}{\partial x^2}\right) &= \left(\tfrac{\partial}{\partial y^1} \quad \tfrac{\partial}{\partial y^2}\right) DF \\ &= \left(\tfrac{\partial}{\partial y^1} \quad \tfrac{\partial}{\partial y^2}\right)\left(\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}\right).\end{aligned}$$

(This is the same type of formula as you used in Worksheet #4, just a slightly different viewpoint – maps instead of coordinate changes.)

So, if $p : (x^1, x^2) = (1, .1)$, then $F(p) = (y^1, y^2) = (10, .1)$ and

$$\begin{aligned}F_* \frac{\partial}{\partial x^1}\Big|_p &= 10\frac{\partial}{\partial y^1}\Big|_{F(p)} \\ F_* \frac{\partial}{\partial x^2}\Big|_p &= -100\frac{\partial}{\partial y^1}\Big|_{F(p)} + \frac{\partial}{\partial y^2}\Big|_{F(p)}.\end{aligned}$$

Locate the basis vectors $\frac{\partial}{\partial y^1}\Big|_{F(p)}$ and $\frac{\partial}{\partial y^2}\Big|_{F(p)}$: