

H/wk 10. Due Friday, April 13.

NAME:

1. Let ω be an r -form in a manifold M^n such that for some r -chain σ in M with $\partial\sigma = 0$ we have $\int_{\sigma} \omega \neq 0$. Prove that ω is not exact.

Solution.

Suppose that ω is exact, so that $\omega = d\alpha$ for some $(r-1)$ -form α . Then by Stokes' Theorem

$$\int_{\sigma} \omega = \int_{\sigma} d\alpha = \int_{\partial\sigma} \alpha = \int_0^0 \alpha = 0,$$

contradicting the assumption that $\int_{\sigma} \omega \neq 0$.

2. Let $T^2 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + w^2 = 1\}$.

Consider the following 1-forms on T^2 : $\omega_1 := \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ and $\omega_2 := \frac{-w}{z^2+w^2}dz + \frac{z}{z^2+w^2}dw$.

Consider the following 1-cubes $\gamma_1 : [0, 1] \rightarrow T^2$ and $\gamma_2 : [0, 1] \rightarrow T^2$ in T^2 :

$$\gamma_1(t) = (\cos 2\pi t, \sin 2\pi t, 1, 0), \quad \gamma_2(t) = (1, 0, \cos 2\pi t, \sin 2\pi t), \quad \text{where } t \in [0, 1].$$

(a) Show that $\partial\gamma_1 = \partial\gamma_2 = 0$.

(b) Compute $\int_{\gamma_1} \omega_1$ and $\int_{\gamma_2} \omega_2$. Conclude that ω_1 and ω_2 are not exact on T^2 .

(c) Consider the chart $(U, \phi = (\theta_1, \theta_2))$ on T^2 where $\phi(U) = (0, 2\pi) \times (0, 2\pi)$ and where $\phi^{-1}(\theta_1, \theta_2) = (\cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2)$. Compute ω_1 and ω_2 in the chart (U, ϕ) , that is, calculate $(\phi^{-1})^*\omega_i$ for $i = 1, 2$. Then verify that $d\omega_1 = 0$ and $d\omega_2 = 0$ in this chart hence, by continuity, $d\omega_1 = 0$ and $d\omega_2 = 0$ on T^2 . Thus ω_1 and ω_2 are closed on T^2 .

(d) Let $c_1, c_2 \in \mathbb{R}$ be such that $c_1^2 + c_2^2 \neq 0$. Show that there exists a linear combination $\sigma = a\gamma_1 + b\gamma_2$ such that $\int_{\sigma} [c_1\omega_1 + c_2\omega_2] \neq 0$ (and hence $c_1\omega_1 + c_2\omega_2$ is not exact since $\partial\sigma = a\partial\gamma_1 + b\partial\gamma_2 = a \cdot 0 + b \cdot 0 = 0$).

Solution.

(a) We have

$$\gamma_1(0) = \gamma_1(1) = (1, 0, 1, 0).$$

Hence $\partial\gamma_1 = \gamma_1(1) - \gamma_1(0) = 0$. Similarly

$$\gamma_2(0) = \gamma_2(1) = (1, 0, 1, 0)$$

and therefore $\partial\gamma_2 = \gamma_2(1) - \gamma_2(0) = 0$.

(b)

$$\begin{aligned} \int_{\gamma_1} \omega_1 &= \int_{[0,1]} \gamma_1^*(\omega_1) = \\ &= \int_{[0,1]} \frac{-\sin 2\pi t}{\cos^2 2\pi t + \sin^2 2\pi t} (-2\pi \sin 2\pi t dt) + \frac{\cos 2\pi t}{\cos^2 2\pi t + \sin^2 2\pi t} (2\pi \cos 2\pi t dt) = \\ &= \int_{[0,1]} 2\pi dt = 2\pi. \end{aligned}$$

A similar computation shows that

$$\int_{\gamma_2} \omega_2 = 2\pi.$$

(c) We have

$$(\phi^{-1})^*\omega_1 = \frac{-\sin\theta_1}{\cos^2\theta_1 + \sin^2\theta_1}(-\sin\theta_1)d\theta_1 + \frac{\cos\theta_1}{\cos^2\theta_1 + \sin^2\theta_1}(\cos\theta_1)d\theta_1 = d\theta_1.$$

A similar computation shows that $(\phi^{-1})^*\omega_2 = d\theta_2$.

Since $dd\theta_1 = 0$ and $dd\theta_2 = 0$, it follows that $d\omega_1 = 0$ and $d\omega_2 = 0$ on U . By continuity this implies that $d\omega_1 = 0$ and $d\omega_2 = 0$ on T^2 , that is, the forms ω_1 and ω_2 are closed on T^2 .

(d) Let $c_1, c_2 \in \mathbb{R}$ be such that $c_1^2 + c_2^2 \neq 0$.

Put $\sigma := c_1\gamma_1 + c_2\gamma_2$. Note that $\gamma_1^*\omega_2 = 0$ and $\gamma_2^*\omega_1 = 0$ and hence $\int_{\gamma_1}\omega_2 = \int_{\gamma_2}\omega_1 = 0$.

Then

$$\begin{aligned} \int_{\sigma} c_1\omega_1 + c_2\omega_2 &= \left(c_1 \int_{\gamma_1} c_1\omega_1 + c_2\omega_2 \right) + \left(c_2 \int_{\gamma_2} c_1\omega_1 + c_2\omega_2 \right) = \\ &= c_1^2 \int_{\gamma_1} \omega_1 + c_1c_2 \int_{\gamma_1} \omega_2 + c_1c_2 \int_{\gamma_2} \omega_1 + c_2^2 \int_{\gamma_2} \omega_2 = \\ &= c_1^2 \cdot 2\pi + c_2^2 \cdot 2\pi = 2\pi(c_1^2 + c_2^2) \neq 0. \end{aligned}$$

We have $\partial\sigma = c_1\partial\gamma_1 + c_2\partial\gamma_2 = 0$.

Hence by Problem 1 the form $c_1\omega_1 + c_2\omega_2$ is not exact on T^2 .

Note: This shows that ω_1, ω_2 are linearly independent in $H_{DR}^1(T^2)$ and so $\dim H_{DR}^1(T^2) \geq 2$. In fact, one can show using a Fourier series argument, that ω_1, ω_2 is a basis of $H_{DR}^1(T^2)$ and that $\dim H_{DR}^1(T^2) = 2$.

3. In the notations of Problem 2, compute

$$\int_{T^2} yzdx \wedge dw.$$

Note.

We can use a good cubulation of T^2 given by $\gamma : [0, 2\pi] \times [0, 2\pi] \rightarrow T^2$, where $\gamma(\theta_1, \theta_2) = (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)$.

$$\begin{aligned} \int_T yzdx \wedge dw &= \int_{[0, 2\pi]^2} \gamma^*(yzdx \wedge dw) = \int_{[0, 2\pi]^2} \sin\theta_1 \cos\theta_2 (-\sin\theta_1)d\theta_1 \wedge \cos\theta_2 d\theta_2 = \\ &= - \int_{[0, 2\pi]^2} \sin^2\theta_1 \cos^2\theta_2 d\theta_1 d\theta_2 = - \int_0^{2\pi} \sin^2\theta_1 d\theta_1 \int_0^{2\pi} \cos^2\theta_2 d\theta_2 = \\ &= - \left(\int_0^{2\pi} \frac{1 - \cos(2\theta_1)}{2} d\theta_1 \right) \left(\int_0^{2\pi} \frac{1 + \cos(2\theta_2)}{2} d\theta_2 \right) = -\pi^2. \end{aligned}$$

4. Let $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Consider a Riemannian metric g on M given by:

$$g|_{(x,y)} \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}, a' \frac{\partial}{\partial x} + b' \frac{\partial}{\partial y} \right) = \frac{aa' + bb'}{y^2},$$

where $(x, y) \in M$ and $a, b, a', b' \in \mathbb{R}$.

(a) Compute the volume form dA on M , corresponding to g , as a multiple of $dx \wedge dy$.

(b) Consider the region $R := \{(x, y) : 0 \leq x \leq 1, y \geq 1\}$ in M . Find the area of this region with respect to dA .

Solution.

(a) We have $dA = \sqrt{EF - G^2} dx \wedge dy$. Here

$$E = g|_{(x,y)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{1}{y^2},$$

$$F = g|_{(x,y)}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{y^2},$$

and

$$G = g|_{(x,y)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0.$$

Hence

$$dA = \sqrt{EF - G^2} dx \wedge dy = \sqrt{\frac{1}{y^4}} dx \wedge dy = \frac{1}{y^2} dx \wedge dy$$

on M .

(b) The area of R is

$$\int_R \frac{1}{y^2} dx dy = \int_0^1 \left(\int_1^\infty \frac{1}{y^2} dy \right) dx = \int_0^1 [-1/y]_1^\infty dx = \int_0^1 dx = 1.$$

5. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a smooth function with $f(x) \geq 0$ for every $x \in [0, 1]$. Let \mathcal{C} be the graph of f in the xy -plane and let M be the surface obtained by rotating \mathcal{C} around the x -axis in \mathbb{R}^3 .

Consider the following 2-cube, giving a good cubulation of M :

$$\gamma : [0, 1] \times [0, 2\pi] \rightarrow M, \gamma(t, \theta) = (t, f(t) \cos \theta, f(t) \sin \theta).$$

We assume that M is given a Riemannian metric that is the restriction of the standard Riemannian metric on \mathbb{R}^3 . Let dA be the resulting volume form on M .

(a) For a point $p = \gamma(t, \theta)$ on M compute the outward normal vector \mathbf{n} to M at p .

Hint. Use the fact that \mathbf{n} is parallel to $\gamma_*\left(\frac{\partial}{\partial t}\right) \times \gamma_*\left(\frac{\partial}{\partial \theta}\right) = \frac{d\gamma}{dt} \times \frac{d\gamma}{d\theta}$.

(b) Compute $dA|_{\gamma(t,\theta)}$ and γ^*dA .

(c) Prove that the surface area of S is equal to

$$\int_0^1 2\pi f \sqrt{1 + (f')^2}.$$

Solution.

(a) We have $\frac{d\gamma}{dt} = (1, f'(t) \cos \theta, f'(t) \sin \theta)$ and $\frac{d\gamma}{d\theta} = (0, -f(t) \sin \theta, f(t) \cos \theta)$. Hence

$$v := \frac{d\gamma}{dt} \times \frac{d\gamma}{d\theta} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & f'(t) \cos \theta & f'(t) \sin \theta \\ 0 & -f(t) \sin \theta & f(t) \cos \theta \end{vmatrix} = f(t)(f'(t), -\cos \theta, -\sin \theta)$$

Hence $\|v\| = f(t)\sqrt{1 + (f'(t))^2}$ and

$$\mathbf{n} = \frac{v}{\|v\|} = \frac{1}{\sqrt{1 + (f'(t))^2}}(f'(t), -\cos \theta, -\sin \theta).$$

(b) Therefore

$$dA|_{\gamma(t,\theta)} = n^1 dy \wedge dz + n^2 dz \wedge dx + n^3 dx \wedge dy = \\ \frac{f'(t)}{\sqrt{1+(f'(t))^2}} dy \wedge dz + \frac{-\cos\theta}{\sqrt{1+(f'(t))^2}} dz \wedge dx + \frac{-\sin\theta}{\sqrt{1+(f'(t))^2}} dx \wedge dy.$$

We have

$$\gamma^* dy \wedge dz = (f'(t) \cos\theta dt - f(t) \sin\theta d\theta) \wedge (f'(t) \sin\theta dt + f(t) \cos\theta d\theta) = \\ = f(t)f'(t) dt \wedge d\theta,$$

$$\gamma^*(dz \wedge dx) = (f'(t) \sin\theta dt + f(t) \cos\theta d\theta) \wedge dt = -f(t) \cos\theta dt \wedge d\theta,$$

and

$$\gamma^*(dx \wedge dy) = dt \wedge (f'(t) \cos\theta dt - f(t) \sin\theta d\theta) = -f(t) \sin\theta dt \wedge d\theta.$$

Therefore

$$\gamma^*(dA) = \frac{1}{\sqrt{1+(f'(t))^2}} (f'(t)^2 f(t) + f(t) \cos^2\theta + f(t) \sin^2\theta) dt \wedge d\theta = \\ f(t) \frac{f'(t)^2 + 1}{\sqrt{1+(f'(t))^2}} dt \wedge d\theta = f(t) \sqrt{1+(f'(t))^2} dt \wedge d\theta$$

(c) The area of S is

$$\int_S dA = \int_{[0,1] \times [0,2\pi]} \gamma^*(dA) = \\ \int_0^1 \left(\int_0^{2\pi} f(t) \sqrt{1+(f'(t))^2} d\theta \right) dt = 2\pi \int_0^1 f(t) \sqrt{1+(f'(t))^2} dt.$$