H/wk 11 (Solutions)

1. Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbb{R}^2$. Let $v = (v^1, v^2)$ and $w = (w^1, w^2)$ be two nonzero vectors in $\mathbb{R}^2$.

Verify that the area of the parallelogram with sides v, w is equal to

$$\left|\det\begin{pmatrix} v^1 & v^2 \\ w^1 & w^2 \end{pmatrix}\right|.$$

Hint. It is easier to prove an equivalent statement saying that the square of the above quantity is equal to the square of the area in question. Also, you can use the fact that if θ is the angle between v and w then

$$\cos\theta = \frac{\langle v, w\rangle}{||v||\ ||w||}.$$

Solution.

The area A of the parallelogram with sides v, w is equal to

$$A = ||v||h = ||v||\ ||w|| \sin\theta$$

where $||v||$ is the length of the base of the parallelogram and $h = ||w|| \sin\theta$ is the hight. Thus

$$A^2 = ||v||^2\ ||w||^2 \sin^2\theta = ||v||^2\ ||w||^2(1 - \cos^2\theta) =$$

$$||v||^2\ ||w||^2(1 - \frac{\langle v, w\rangle^2}{||v||^2\ ||w||^2}) = ||v||^2\ ||w||^2 - \langle v, w\rangle^2 =$$

$$((v^1)^2 + (v^2)^2)((w^1)^2 + (w^2)^2) - (v^1w^1 + v^2w^2)^2 =$$

$$(v^1w^1)^2 + (v^1w^2)^2 + (v^2w^1)^2 + (v^2w^2)^2 - (v^1w^1)^2 - (v^2w^2)^2 - 2v^1w^1v^2w^2 =$$

$$(v^1w^2)^2 + (v^2w^1)^2 - 2v^1w^1v^2w^2 = (v^1w^2 - v^2w^1)^2 = \det{}^2\begin{pmatrix} v^1 & v^2 \\ w^1 & w^2 \end{pmatrix}.$$

Hence

$$A = \left|\det\begin{pmatrix} v^1 & v^2 \\ w^1 & w^2 \end{pmatrix}\right|,$$

as required.

2. Consider the cylinder $M := \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1, -1 \leq x \leq 1\}$ with the orientation given by the outward unit normal $\mathbf{n} = (0, y, z)$. Thus M is an oriented 2-manifold-with-boundary in $\mathbb{R}^3$.

(a) Sketch M in the xyz-space. Indicate orientation on M by showing the rotation in the positive (in the sense of the orientation on M) direction near the point $(0, 0, 1)$ on M.

(b) Indicate the directions on the two components of the boundary ∂M corresponding to the induced orientation on ∂M from M.

(c) Let $p = (1, y, z) \in \partial M$ be a point in the "right-hand" component of ∂M. Find the unit tangent vector T at p to ∂M such that T is positively oriented with respect to the orientation on ∂M induced from M.

Do the same for a point $p = (-1, y, z) \in \partial M$ in the "left-hand" component of ∂M.

(d) Consider the vector field $F = (e^{x^4 y^4 z^4}, zx, -yx)$ on $\mathbb{R}^3$.
Compute

$$\int_M \langle \operatorname{curl} F, \mathbf{n} \rangle \, dA$$

Solution.
(a) (b)

FIGURE 1. Problem 2, parts (a)(b)

(c) At the point $p = (1, y, z)$ on ∂M (where $y^2 + z^2 = 1$) there are two unit tangents to the circle ∂M: $T_1 = (0, z, -y)$ and $T_2 = -T_1 = (0, -z, y)$. The vector $\nu = (1, 0, 0)$ is the tangent vector to M at p that is outward unit normal to ∂M at p.

Thus T_i is positively oriented with respect to the induced orientation on ∂M if and only if ν, T_i is a positive basis of M_p, that is, if and only if $\mathbf{n}, \nu, T_i$ is a positive basis of $\mathbb{R}^3$. For T_1 the matrix with rows $\mathbf{n}, \nu, T_1$ is:

$$\det \begin{pmatrix} 0 & y & z \\ 1 & 0 & 0 \\ 0 & z & -y \end{pmatrix} = y^2 + z^2 = 1 > 0$$

Therefore $T_1 = (0, z, -y)$ is the positively oriented unit tangent to ∂M at $p = (1, y, z)$.

(d) By the Divergence Theorem

$$\int_M \langle \operatorname{curl} F, \mathbf{n} \rangle \, dA = \int_{\partial M} \langle F, T \rangle \, ds$$

where T is a positively oriented unit tangent to ∂M. The boundary ∂M consists of two component circles: $\partial_R M = \{(1, y, z) : y^2 + z^2 = 1\}$ and

$\partial_L M = \{(-1, y, z) : y^2 + z^2 = 1\}$. We have

$$\int_{\partial M} \langle F, T \rangle \; ds = \int_{\partial_R M} \langle F, T \rangle \; ds + \int_{\partial_L M} \langle F, T \rangle \; ds.$$

For the circle $\partial_R M$ we have

$$\int_{\partial_R M} \langle F, T \rangle \; ds = \int_{\partial_R M} \langle (e^{x^4 y^4 z^4}, zx, -yx), (0, z, -y) \rangle \; ds =$$

$$\int_{\partial_R M} z^2 x - y^2 x \; ds = \int_{\partial_R M} x(z^2 + y^2) \; ds =$$

$$\int_{\partial_R M} \; ds = 2\pi.$$

where we use the fact that $x = 1$ and $z^2 + y^2 = 1$ on $\partial_R M$.

A similar computation for $\partial_L M$ gives:

$$\int_{\partial_L M} \langle F, T \rangle \; ds = \int_{\partial_R M} \langle (e^{x^4 y^4 z^4}, zx, -yx), (0, -z, y) \rangle \; ds =$$

$$\int_{\partial_R M} -z^2 x - y^2 x \; ds = \int_{\partial_R M} (-x)(z^2 + y^2) \; ds =$$

$$\int_{\partial_R M} \; ds = 2\pi.$$

where we use the fact that $x = -1$ and $z^2 + y^2 = 1$ on $\partial_L M$.

Hence

$$\int_M \langle \operatorname{curl} F, \mathbf{n} \rangle \; dA = 2\pi + 2\pi = 4\pi.$$

3. Let $\mathbb{B}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ be the unit ball in $\mathbb{R}^3$.

(a) Find a vector field F on $\mathbb{R}^3$ such that $\langle F, \mathbf{n} \rangle = const$ on $\mathbb{S}^2$ (where $\mathbf{n}$ is the outward unit normal on $\mathbb{S}^2$) and $divF = const$ on $\mathbb{R}^3$.

(b) Prove that $3 \cdot vol(\mathbb{B}^3) = area(\mathbb{S}^2)$ using the Divergence Theorem, without computing these quantities precisely.

Solution.

(a) Take $F = (x, y, z)$. Then $divF = 3$ on $\mathbb{R}^3$. Since on $\mathbb{S}^2$ the outward unit normal at $p = (x, y, z) \in \mathbb{S}^2$ is $\mathbf{n} = (x, y, z)$, we have $\langle F, \mathbf{n} \rangle = x^2 + y^2 + z^2 = 1$.

(b) By the Divergence Theorem, with $F = (x, y, z)$ as in part (a), we have

$$\int_{\mathbb{B}^3} divF \; dV = \int_{\mathbb{S}^2} \langle F, \mathbf{n} \rangle \; dA$$

and hence

$$3 \; vol(\mathbb{B}^3) = \int_{\mathbb{B}^3} 3 \; dV = \int_{\mathbb{S}^2} \; dA = area(\mathbb{S}^2).$$

4. Let $a, b, c > 0$ be three fixed positive numbers.

Consider the ellipsoid surface

$$M := \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$$

and the solid ellipsoid

$$N := \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}.$$

Thus N is a 3-manifold-with-boundary in $\mathbb{R}^3$ and $M = \partial N$. We give N the standard orientation from $\mathbb{R}^3$ and endow $M = \partial N$ with the induced orientation from N.

(a) For a general point $p = (x,y,z) \in M$ compute the outward unit normal $\mathbf{n}$ to M at p.

(b) Compute the area form dA on M in terms of x, y, z.

Consider the chart (U, f) on M with $f^{-1} = h : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ where

$$h(\theta, \phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$$

and $U = h\left((0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})\right)$

(c) Verify that (U, f) is an orienting chart for M, that is, that $\frac{\partial}{\partial\theta}|_p, \frac{\partial}{\partial\phi}|_p$ is a positively oriented basis for every $p \in U$.

(This can be done, for example, by checking that $\frac{\partial h}{\partial\theta} \times \frac{\partial h}{\partial\phi}$ is a positive scalar multiple of $\mathbf{n}|_{h(\theta,\phi)}$).

(d) Compute the area form dA in the chart (U, f).

Solution.

(a) Put $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, $F : \mathbb{R}^3 \to \mathbb{R}$. At a point $p = (x,y,z) \in M$ we have

$$grad\ F = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{z^2})$$

and

$$||grad\ F|| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} = 2\frac{\sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}}{a^2b^2c^2}.$$

Therefore the outward unit normal at p is

$$\mathbf{n} = \frac{grad\ F}{||grad\ F||} =$$

$$\frac{1}{\sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}}(xb^2c^2,\ ya^2c^2,\ za^2b^2).$$

(b) We have

$$dA = n^1 dy \wedge dz + n^2 dz \wedge dx + n^3 dx \wedge dy =$$

$$\frac{1}{\sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}}\left(xb^2c^2 dy \wedge dz + ya^2c^2 dz \wedge dx + za^2b^2 dx \wedge dy\right).$$

(c) We have $\frac{\partial h}{\partial\theta} = (-a\sin\theta\cos\phi, b\cos\theta\cos\phi, 0)$ and

$\frac{\partial h}{\partial \phi} = (-a\cos\theta\sin\phi, -b\sin\theta\sin\phi, c\cos\phi)$.

Therefore

$$\frac{\partial h}{\partial \theta} \times \frac{\partial h}{\partial \phi} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix} =$$

$$(bc\cos\theta\cos^2\phi, ac\sin\theta\cos^2\phi, ab\cos\phi\sin\phi).$$

Recall that

$$h(\theta, \phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$$

and that

$$\mathbf{n} = \frac{1}{\sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}}(xb^2c^2,\ ya^2c^2,\ za^2b^2),$$

Put

$$\mathbf{m} = (xb^2c^2,\ ya^2c^2,\ za^2b^2) = \sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}\ \mathbf{n}.$$

Hence

$$\frac{\partial h}{\partial \theta} \times \frac{\partial h}{\partial \phi} = \frac{\cos\phi}{abc}\ \mathbf{m} = \frac{\cos\phi\ \sqrt{x^2b^4c^4 + y^2a^4c^4 + z^2a^4b^4}}{abc}\ \mathbf{n}.$$

Recall that $\phi \in (-\pi/2, \pi/2)$ and $\cos(\phi) > 0$ for $-\pi/2 < \phi < \pi/2$. Thus for $-\pi/2 < \phi < \pi/2$ the vector $\frac{\partial h}{\partial \theta} \times \frac{\partial h}{\partial \phi}$ is a positive scalar multiple of $\mathbf{n}$ and hence (U, f) is an orienting chart for M, as required.

(d) We have

$$E = \langle \frac{\partial h}{\partial \theta}, \frac{\partial h}{\partial \theta} \rangle = a^2\sin^2\theta\cos^2\phi + b^2\cos^2\theta\cos^2\phi,$$

$$F = \langle \frac{\partial h}{\partial \theta}, \frac{\partial h}{\partial \phi} \rangle = (a^2 - b^2)\cos\theta\sin\theta\cos\phi\sin\phi,$$

and

$$G = \langle \frac{\partial h}{\partial \phi}, \frac{\partial h}{\partial \phi} \rangle = a^2\cos^2\theta\sin^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\phi.$$

Therefore un the chart (U, f) we have

$$dA = \sqrt{EF - G^2}\ d\theta \wedge d\phi =$$

$$\big[(a^2\sin^2\theta\cos^2\phi + b^2\cos^2\theta\cos^2\phi)(a^2 - b^2)\cos\theta\sin\theta\cos\phi\sin\phi -$$

$$(a^2\cos^2\theta\sin^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\phi)^2\big]^{1/2}\ d\theta \wedge d\phi$$