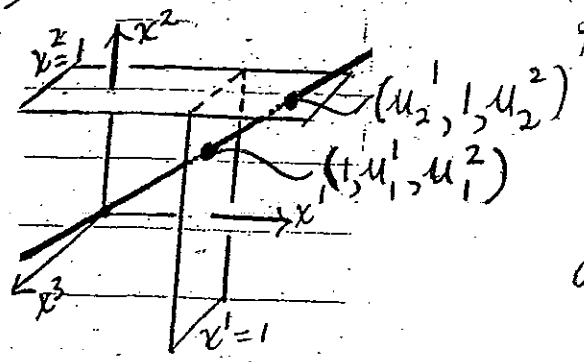
MTH. 581 #2 DUE. NAME: answers

1 P^2 = all lines through 0 in $\mathbb{R}^3$. Put an atlas with 3 charts on P^2, as in Worksheet #1:



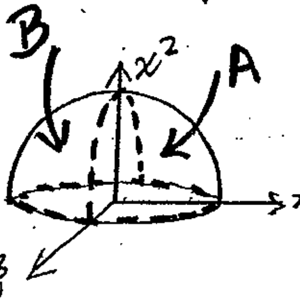
$(U_1, \phi_1=(u_1^1,u_1^2)), (U_2,\phi_2=(u_2^1,u_2^2)), (U_3,\phi_3=(u_3^1,u_3^2))$.

Then $u_2^1 = \frac{1}{u_1^1}$, $u_2^2 = \frac{u_1^2}{u_1^1}$.

a) Calculate the determinant of $D(\phi_2\circ\phi_1^{-1}) = \frac{\partial(u_2^1,u_2^2)}{\partial(u_1^1,u_1^2)}$

$$\det\begin{pmatrix} -1/(u_1^1)^2 & 0 \\ -u_1^2/(u_1^1)^2 & 1/u_1^1 \end{pmatrix} = -\frac{1}{(u_1^1)^3}$$ 2

b)



Carefully show on the sketch, and describe in words, the points on the upper hemisphere that lie on lines in $U_1 \cap U_2$.

Points NOT on x^2x^3 or x^1x^3 planes 2

c) Discuss the sign of the determinant of $D(\phi_2\circ\phi_1^{-1})$ for $U_1\cap U_2$.

+ on B

− on A 2

d) By switching the coordinates of one (or more) of your charts can you make all $\det D(\phi_j\circ\phi_i^{-1}) > 0$? Explain, using c).

NO. Switching a pair of coordinates in ϕ_1 or ϕ_2 will give "−" on B and "+" on A. There is no way to get an oriented atlas. 1/7

MA. 381 # 2, P 2 NAME: answers

2/ P^3 = all lines through 0 in $\mathbb{R}^4$. Put 4 charts on P^3 as follows: For $1 \le i \le 4$, let U_i = all lines through 0 that intersect the 3-plane $x^i = 1$. As before, let (u_i^1, u_i^2, u_i^3) be the other 3 coordinates of the intersection point.

(a) For $k = 2, 3, 4$, write u_k^1, u_k^2, u_k^3 as functions of u_1^1, u_1^2, u_1^3.

$u_2^1 = 1/u_1^1$ $\qquad u_3^1 = 1/u_1^2$ $\qquad u_4^1 = 1/u_1^3$

$u_2^2 = u_1^2/u_1^1$ $\qquad u_3^2 = u_1^1/u_1^2$ $\qquad u_4^2 = u_1^1/u_1^3$

$u_2^3 = u_1^3/u_1^1$ $\qquad u_3^3 = u_1^3/u_1^2$ $\qquad u_4^3 = u_1^2/u_1^3$

Explain:

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(b) Calculate $\det D(\phi_k \circ \phi_1^{-1}) = \det \dfrac{\partial(u_k^1, u_k^2, u_k^3)}{\partial(u_1^1, u_1^2, u_1^3)}$ for $U_1 \cap U_k$:

On $U_1 \cap U_2$: $-1/(u_1^1)^4$

On $U_1 \cap U_3$: $+1/(u_1^2)^4$

On $U_1 \cap U_4$: $-1/(u_1^3)^4$

2

Explain:

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ⓒ Show that you can make all $\det D(\phi_k \circ \phi_1^{-1}) > 0$, by switching a pair of coordinates in one or more of your charts.

Switching a pair in (U_2, ϕ_2) and in (U_4, ϕ_4) will work.

(Switching a pair in (U_1, ϕ_1) will NOT.)

②

ⓓ Note that $D(\phi_j \circ \phi_i^{-1}) = D(\phi_j \circ \phi_1^{-1})\, D(\phi_1 \circ \phi_i^{-1})$

$$= D(\phi_j \circ \phi_1^{-1})\, D(\phi_i \circ \phi_1^{-1})^{-1}.$$

Use this to show you can make $\det D(\phi_j \circ \phi_i^{-1}) > 0$ on $U_i \cap U_j \cap U_1$, for all i, j.

$\det\left[D(\phi_i \circ \phi_1^{-1})^{-1}\right] = \left[\det D(\phi_i \circ \phi_1^{-1})\right]^{-1}$

$\therefore \det\left[D(\phi_j \circ \phi_i^{-1})\right] = \det\left[D(\phi_j \circ \phi_1^{-1})\right] \det\left[D(\phi_i \circ \phi_1^{-1})\right]^{-1}$

Left side is + because right side is (+)(+) by part ⓑ. ②

But in P^3 (as opposed to P^2!), the sign of $D(\phi_j \circ \phi_i^{-1})$ is the same on all the components of $U_i \cap U_j$, so you have made all $D(\phi_j \circ \phi_i^{-1}) > 0$ everywhere...

CONGRATULATIONS!!

You just proved that $\boxed{P^3 \text{ IS ORIENTABLE.}}$

ⓔ You have enough information now to make a good guess:

P^n is orientable if n is odd

not orientable if n is even

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