H/wk 7 Solutions

1. a)

$$(W')^j_i = \sum_{k,\ell} \frac{\partial y^j}{\partial x^k}\frac{\partial x^\ell}{\partial y^i} W^k_\ell = \frac{\partial y^j}{\partial x^1}\frac{\partial x^2}{\partial y^i} W^1_2 =$$

$$= \frac{\partial y^j}{\partial x^1}\frac{\partial x^2}{\partial y^i}\cdot (x^1)^2 x^2.$$

b) $y^1 = 2x^1$, $y^2 = x^1x^2 \Rightarrow x^1 = \frac{1}{2}y^1$

$x^2 = y^2/x^1 = 2\frac{y^2}{y^1}$

$$\frac{\partial y^1}{\partial x^1} = 2 \quad \frac{\partial y^1}{\partial x^2} = 0$$

$$\frac{\partial y^2}{\partial x^1} = x^2 = 2\frac{y^2}{y^1}, \quad \frac{\partial y^2}{\partial x^2} = x^1 = y^1/2$$

$$\frac{\partial x^1}{\partial y^1} = 1/2 \quad \frac{\partial x^1}{\partial y^2} = 0 \quad \frac{\partial x^2}{\partial y^1} = -\frac{2y^2}{(y^1)^2} \quad \frac{\partial x^2}{\partial y^2} = \frac{2}{y^1}$$

Hence

$$(W')^1_1 = \frac{\partial y^1}{\partial x^1}\frac{\partial x^2}{\partial y^1}(x^1)^2\cdot(x^2) = 2\cdot\left(-\frac{2y^2}{(y^1)^2}\right)\cdot\frac{(y^1)^2}{4}\cdot 2\frac{y^2}{y^1} =$$

$$= -2\frac{(y^2)^2}{y^1}$$

$$(W')^1_2 = \frac{\partial y^1}{\partial x^1}\frac{\partial x^2}{\partial y^2}\cdot(x^1)^2\cdot x^2 = 2\cdot\frac{2}{y^1}\cdot\frac{(y^1)^2}{4}\cdot\frac{2y^2}{y^1} = 2y^2$$

$$(W')^2_1 = \frac{\partial y^2}{\partial x^1}\cdot\frac{\partial x^2}{\partial y^1}\cdot(x^1)^2 x^2 = 2\frac{y^2}{y^1}\cdot\frac{-2y^2}{(y^1)^2}\cdot\frac{(y^1)^2}{4}\cdot\frac{2y^2}{y^1} =$$

$$= -2\frac{(y^2)^3}{(y^1)^2}$$

$$(W')^2_2 = \frac{\partial y^2}{\partial x^1}\frac{\partial x^2}{\partial y^2}(x^1)^2 x^2 =$$

$$= 2\frac{y^2}{y^1}\cdot\frac{2}{y^1}\cdot\frac{(y^1)^2}{4}\cdot\frac{2y^2}{y^1} = 2\frac{(y^2)^2}{y^1}$$

c) $W(p)\left(dy^1+dy^2, \frac{\partial}{\partial y^1}\right)=$

$= W(p)\left(dy^1, \frac{\partial}{\partial y^1}\right) + W(p)\left(dy^2, \frac{\partial}{\partial y_1}\right) =$

$= W'(p)^1_1 + W'(p)^2_1 =$

$= (-2)\frac{(y_2)^2}{y^1} + (-2)\frac{(y^2)^3}{(y^1)^2} =$

$=(-2)\cdot \frac{1^2}{5} + (-2)\cdot \frac{1^3}{5^2} = -\frac{12}{25}$ since $\Psi(p)=(5,1)$, $y^1_{(p)} = 5$, $y^2_{(p)} = 1$

2. a) $L(\gamma) = \int_0^1 \sqrt{g|_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\, dt$

$L(\gamma_1^B) = \int_0^1 \sqrt{g|_{(\alpha\circ\gamma)(t)}(\dot\gamma_1(t),\dot\gamma_1(t))}\, dt$

We claim that $\dot\gamma_1(t) = \alpha_* \dot\gamma(t)$. If that is so, then

$g|_{\alpha(\gamma(t))}(\dot\gamma_1(t),\dot\gamma_1(t)) = g_{\alpha(\gamma(t))}(\alpha_*\dot\gamma(t), \alpha_*\dot\gamma(t)) =$

$= g|_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))$ and hence $L(\gamma)=L(\gamma_1)$

since α is an isometry

Recall that by definition $\dot\gamma(t) = \gamma_*\left(\frac{\partial}{\partial t}\right)$

$\dot\gamma_1(t) = \gamma_{1*}\left(\frac{\partial}{\partial t}\right)$

Thus $\dot\gamma_1(t) = (\alpha\circ\gamma)_*\left(\frac{\partial}{\partial t}\right) = (\alpha_*\circ\gamma_*)\left(\frac{\partial}{\partial t}\right) = \alpha_*\left(\gamma_*\frac{\partial}{\partial t}\right) = \alpha_*\dot\gamma(t)$ as required.

b) We have $\alpha(x,y) = (\lambda x, \lambda y) \Rightarrow$

$$\alpha_*\left(\frac{\partial}{\partial x}\Big|_{(x,y)}\right) = \lambda \frac{\partial}{\partial x}\Big|_{(\lambda x, \lambda y)} \quad \alpha_*\left(\frac{\partial}{\partial y}\Big|_{(x,y)}\right) = \lambda \frac{\partial}{\partial y}\Big|_{(\lambda x, \lambda y)}$$

Let $\vec{v} = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}$, $\vec{w} = w^1 \frac{\partial}{\partial x} + w^2 \frac{\partial}{\partial y}$ be arbitrary tangent vectors at $p = (x,y)$ to M.

Then $g|_{(x,y)}(\vec{v}, \vec{w}) = \dfrac{v^1 w^1 + v^2 w^2}{y^2}$ by definition of g.

We have

$$g|_{\alpha(x,y)}(\alpha_* \vec{v}, \alpha_* \vec{w}) = g|_{(\lambda x, \lambda y)}\left(\lambda v^1 \frac{\partial}{\partial x} + \lambda v^2 \frac{\partial}{\partial y}, \lambda w^1 \frac{\partial}{\partial x} + \lambda w^2 \frac{\partial}{\partial y}\right)$$

$$\underset{\substack{\downarrow \\ \text{by definition of } g}}{=} \frac{\lambda v^1 \cdot \lambda w^1 + \lambda v^2 \cdot \lambda w^2}{(\lambda y)^2} = \frac{v^1 w^1 + v^2 w^2}{y^2} = g|_{(x,y)}(\vec{v}, \vec{w}),$$

as required.

c) $\gamma(t) = (0, 1+t)$, $t \in [0, T]$,

$$\dot{\gamma}(t) = 0 \frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial y}\Big|_{\gamma(t)}$$

$$g|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = g|_{(0,1+t)}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{(1+t)^2}$$

$$L(\gamma) = \int_0^T \sqrt{g|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt = \int_0^T \frac{1}{(1+t)}\, dt =$$

$$= \Big[\ln(1+t)\Big]_0^T = \ln(1+T).$$

a) $V = x\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}$

$\omega|_p(\vec{w}) := g|_p(V(p), \vec{w})$. For $p = (x,y)$

$$\omega|_{(x,y)}\left(\frac{\partial}{\partial x}\right) = g|_{(x,y)}\left(x\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{2}{y^2}$$

$$\omega|_{(x,y)}\left(\frac{\partial}{\partial y}\right) = g|_{(x,y)}\left(x\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{x}{y^2}$$

Thus $\omega = \frac{2}{y^2}dx + \frac{x}{y^2}dy$.