The volume form.

1. Orientation on vector spaces and manifolds.

(1) Let V be a vector space of dimension $n \geq 1$. Recall that an *orientation* on V is specified by choosing a particular basis $e_1, \ldots, e_n$ of V. If $v_1, \ldots, v_n$ is any other basis of V, this basis is declared *positive* if the transition matrix A has $det(A) > 0$ and *negative* if $det(A) < 0$.

Here $A = (v_i^j)_{ij}$, where $v_i = \sum_{j=1}^n v_i^j e_j$. Note that if $v_1 = e_1, \ldots v_n = e_n$, then $A = I_n$ is the identity matrix and hence $e_1, \ldots, e_n$ is a positive basis.

(2) Let M^n be an n-manifold. An *orientation* of M can be viewed as a continuous (with respect to varying p) choice of orientations on all the tangent spaces M_p, where $p \in M$.

Let $(U_i, \phi_i)_i$ be an orienting atlas on M (that is the Jacobians of all the transition maps $\phi_i \circ \phi_j^{-1}$ have positive determinants). For $p \in M$ this atlas defines an orientation on M_p by declaring that the basis $\frac{\partial}{\partial x_i^1}|_p, \ldots, \frac{\partial}{\partial x_i^n}|_p$ is a positive basis of M_p, where $p \in (U_i, \phi_i = (x_i^1, \ldots, x_i^n))$.

Conversely, suppose that we have specified a a continuous (with respect to varying p) choice of orientations on all the tangent spaces M_p, where $p \in M$. We can construct an orienting atlas on M as follows. Start with any atlas $(U_i, \phi_i = (x_i^1, \ldots, x_i^n))_i$ with connected sets U_i. For every U_i choose $p \in U_i$ and check whether $\frac{\partial}{\partial x_i^1}|_p, \ldots, \frac{\partial}{\partial x_i^n}|_p$ is a positive basis of M_p. If yes, keep (U_i, ϕ_i) without change. If not, interchange x_i^1 and x_i^2 in ϕ_i. Do this for every i. The resulting collection of charts is an orienting atlas on M.

2. The volume form on a vector space. Let V be a vector space of dimension $n \geq 1$ with a chosen orientation. Let $\langle\ ,\ \rangle$ be a positive-definite inner product on V. The *volume form* on V corresponding to this choice of an orientation and to $\langle\ ,\ \rangle$ is the unique n-form ω on V such that for any positive basis $v_1, \ldots, v_n$ of V we have

$$(\dagger) \qquad \omega = \sqrt{|g|}\, v_1^* \wedge \cdots \wedge v_n^*,$$

here $|g|$ is the determinant of the $n \times n$ symmetric matrix g with $g_{ij} = \langle v_i, v_j \rangle$.

Facts.

(1) The form defined by ($\dagger$) does not depend on the choice of a positive basis $v_1, \ldots, v_n$ of V.

(2) If $e_1, \ldots, e_n$ is a positive orthonormal basis of V then $\omega = e_1^* \wedge \cdots \wedge e_n^*$. (To see this, just apply the formula ($\dagger$) in this case.

(3) let $e_1, \ldots, e_n$ be a positive orthonormal basis of V. Let $v_1, \ldots, v_n$ be any other basis of V with $v_i = \sum_{j=1}^n v_i^j e_j$. Let A be the transition matrix $A = (v_i^j)_{ij}$.

Then

$$\omega(v_1, \ldots, v_n) = |det(A)|.$$

Thus $\omega(v_1, \ldots, v_n)$ is equal to the volume of the n-dimensional box with sides $v_1, \ldots, v_n$ in V, computed using the standard formulas of Euclidean geometry.

(4) Let $dim(V) = 1$. Let $v \in V, v \neq 0$, so that $\{v\}$ is a basis of V. Then g is a 1×1-matrix with the entry $g_{1,1} = \langle v, v\rangle = ||v||^2$. Hence $\omega = ||v||v^*$ if v is positive (in the sense of the orientation on V) and $\omega = -||v||v^*$ if v is negative. In particular if $e \in V$ is a positive unit vector, then $\omega = e^*$. Note that if $w \in V$ is any other vector, then $w = ce$, where $c = \epsilon||w||$, where $\epsilon = 1$ if $w \neq 0$ is a positive vector, $\epsilon = -1$ if $w \neq 0$ is negative and $\epsilon = 0$ if $w = 0$. then $\omega(w) = e^*(ce) = c = \epsilon||w||$. Thus the volume form $\omega(w)$ computes the "signed length" of a w in this case.

3. The volume form on a manifold. Let M^n be an oriented n-manifold with a Riemannian metric g. The *volume form* $dvol_g$ is an n-form on M such that if $(U, \phi = (x^1, \dots, x^n))$ is a chart from an orienting atlas then in this chart

$$dvol_g = \sqrt{|g|}\; dx^1 \wedge \dots dx^n$$

where g is the $n \times n$-matrix with $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

4. The volume form on a surface. Let M^2 be an oriented surface with a Riemannian metric g and with an orienting chart $(U, \phi = (x^1, x^2))$. Then in this chart

$$dA = dvol_g = \sqrt{EF - G^2} dx^1 \wedge dx^2,$$

where $E = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})$, $F = g(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2})$ and $G = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$.

5. Outward unit normals for a surface in $\mathbb{R}^3$. Let M^2 be a surface in $\mathbb{R}^3$, where $\mathbb{R}^3$ is considered with coordinates (x, y, z).

Here for $p \in M$ we identify the tangent space M_p with the set of geometric tangent vectors to M at p in $\mathbb{R}^3$, so that $M_p \leq \mathbb{R}^3$ is a 2-dimensional linear subspace.

Recall that if $(U, \phi = (x^1, x^2))$ is a chart on M and $\psi = \phi^{-1} : U \to \mathbb{R}^3$, $\psi = (\psi_1 = x(x^1, x^2), \psi_2 = y(x^1, x^2), \psi_3 = z(x^1, x^2))$ then an abstract tangent vector $a\frac{\partial}{\partial x^1}|_p + b\frac{\partial}{\partial x^2}|_p \in M_p$ is identified with the geometric tangent vector

$$\frac{\partial(x, y, z)}{\partial(x^1, x^2)}|_p \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\frac{\partial x}{\partial x^1}|_p + b\frac{\partial x}{\partial x^2}|_p \\ a\frac{\partial y}{\partial x^1}|_p + b\frac{\partial y}{\partial x^2}|_p \\ a\frac{\partial z}{\partial x^1}|_p + b\frac{\partial z}{\partial x^2}|_p. \end{bmatrix} \in \mathbb{R}^3$$

for any $a, b \in \mathbb{R}$.

(1) If M^2 is an oriented surface, then at every point in $p \in M$ one can define the *outward unit normal* $\mathbf{n}(p) \in \mathbb{R}^3$. Note that there are precisely two (opposite) vectors of length 1 in $\mathbb{R}^3$ that are perpendicular to the 2-dimensional subspace $M_p \leq \mathbb{R}^3$. The outward unit normal $\mathbf{n}(p)$ is the unique unit normal to M_p with the property that if $v_1, v_2 \in M_p \leq \mathbb{R}^3$ is a positive basis of M_p then $\mathbf{n}(p), v_1, v_2$ is a positive basis of $\mathbb{R}^3$ with respect to the standard orientation on $\mathbb{R}^3$.

Specifically, in this case

$$\mathbf{n}(p) = \frac{v_1 \times v_2}{||v_1 \times v_2||}$$

(2) Suppose M^2 is an oriented surface and $(U, \phi = (x^1, x^2))$ is an orienting chart with $\psi = \phi^{-1} = (\psi_1 = x(x^1, x^2), \psi_2 = y(x^1, x^2), \psi_3 = z(x^1, x^2))$

Then for $p \in U$ the geometric tangent corresponding to $\frac{\partial}{\partial x^1}|_p$ is $\frac{\partial \psi}{\partial x^1}|_p \in \mathbb{R}^3$ and the geometric tangent corresponding to $\frac{\partial}{\partial x^2}$ is $\frac{\partial \psi}{\partial x^2}|_p \in \mathbb{R}^3$. Hence we can compute the outward unit normal as:

$$\mathbf{n}(p) = \frac{\frac{\partial \psi}{\partial x^1}|_p \times \frac{\partial \psi}{\partial x^2}|_p}{||\frac{\partial \psi}{\partial x^1}|_p \times \frac{\partial \psi}{\partial x^2}|_p||}.$$

(3) Let $f : \mathbb{R}^3$ is a smooth function such that for every $p \in M$ with

$$M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}$$

we have $grad\ f_p \neq 0$ (so that M is a 2-manifold).

Let

$$N = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) \leq 0\}.$$

Then N is a 3-manifold-with-boundary in $\mathbb{R}^3$ and $\partial N = M$. We give N the same orientation as in $\mathbb{R}^3$ and we give $M = \partial N$ the induced orientation. Then for any $p \in M$ we can compute the outward unit normal as

$$\mathbf{n}(p) = \frac{grad\ f|_p}{||grad\ f|_p||}.$$

(4) Let $M^2 \subseteq \mathbb{R}^3$ be a surface. Suppose that for every $p \in M$ we have chosen a unit vector $\mathbf{n}(p) \in \mathbb{R}^3$ that is perpendicular to M_p and such that $\mathbf{n}(p)$ varies continuously with p. This defines an orientation on M. Namely, for $p \in M$ and a basis $v_1, v_2, \in M_p$ we declare that v_1, v_2 is a positive basis of M_p if and only if $\mathbf{n}(p), v_1, v_2$ is a positive basis of $\mathbb{R}^3$ (that is, if and only if $\mathbf{n}(p) = cv_1 \times v_2$ where $c > 0$). This determines an orientation on M_p with respect to which $\mathbf{n}(p)$ is the outward unit normal.

6. The volume form for a surface in $\mathbb{R}^3$. Let $M^2 \subseteq \mathbb{R}^3$ be an oriented surface with an outward unit normal $\mathbf{n} = (n^1, n^2, n^3)$. We endow M with a Riemannian metric g that is the restriction to M_p of the standard inner product in $\mathbb{R}^3$. That is, if $v_1 = (v_1^1, v_1^2, v_1^3)$ and $v_2 = (v_2^1, v_2^2, v_2^3)$ are elements of $M_p \leq \mathbb{R}^3$, we put

$$g|_p(v_1, v_2) = \langle v_1, v_2 \rangle = v_1^1 v_2^1 + v_1^2 v_2^2 + v_1^3 v_2^3.$$

Then the volume form dA on M is:

$$dA = n^1 dy \wedge dz + n^2 dz \wedge dx + n^3 dx \wedge dy.$$

Moreover,

$$n^1 dA = dy \wedge dz,\ n^2 dA = dz \wedge dx,\ n^3 dA = dx \wedge dy$$

on M.

7. The volume form for a curve in $\mathbb{R}^3$. Let $M^1 \subseteq \mathbb{R}^3$ be an oriented 1-manifold (that is, a curve in $\mathbb{R}^3$ with a chosen direction). For every point $p \in M$ there are exactly two (opposite) unit tangent vectors to M at p. Let $T|_p = (T_1|_p, T_2|_p, T_3|_p) \in \mathbb{R}^3$ be the unique unit tangent vector to M at p

which is positively oriented with respect to the orientation on M. We endow M with a Riemannian metric g that is the restriction to M_p of the standard inner product in $\mathbb{R}^3$. Denote by ds the volume form on M corresponding to g. Then

$$ds = T^1 dx + T^2 dy + T^3 dz$$

on M. Moreover,

$$T^1 ds = dx,\ T^2 ds = dy,\ T^3 ds = dz$$

on M.